UNIT – III
FLUID FLOW CONCEPTS

KINEMATICS OF FLOW:

"Branch of science which deals with motion of particles without considering the forces causing the motion."

- The velocity at any point in a flow filed at any time is studied.
- Once the velocity is determined, then the pressure distribution and hence force acting on the fluid can be determined

In kinematics velocity and acceleration of the fluid is discussed.

METHODS OF DESCRIBING FLUID MOTION:

1. Lagrangians.
2. Eulerian

- In *"Lagrangian method",* a single **fluid particle is followed** during its motion and its velocity, acceleration, density, etc., are described.
- In *"Eulerian method",* the velocity, acceleration, density, etc., are described **at a point** in flow fluid.

TYPES OF FLUID FLOW:

The fluid flow is classified as:

1. Steady and Unsteady flows.
2. Uniform and Non-uniform flows.
3. Laminar and turbulent flows.
4. Compressible and incompressible flows.
5. Rotational and irrotational flows, and
6. One, Two and Three dimensional flows.

1. Steady and Unsteady flows:

Steady Flow : Flow in which the fluid characteristics like velocity, pressure, density etc at a point do not change with "time".

$$\frac{\partial v}{\partial t}_{x_o y_o z_o} = 0; \frac{\partial p}{\partial t}_{x_o y_o z_o} = 0; \frac{\partial \rho}{\partial t}_{x_o y_o z_o} = 0$$

Unsteady flow: Flow in which the fluid characteristics like velocity, pressure, density etc at a point changes with "time".

$$\frac{\partial v}{\partial t}_{x_o y_o z_o} \neq 0; \frac{\partial p}{\partial t}_{x_o y_o z_o} \neq 0; \frac{\partial \rho}{\partial t}_{x_o y_o z_o} \neq 0$$

2. Uniform and Non-uniform flow:

Uniform flow: Flow in which the fluid characteristics like velocity, pressure, density etc at a point do not change with "space".

$$\frac{\partial v}{\partial s}_{x_o y_o z_o} = 0; \frac{\partial p}{\partial s}_{x_o y_o z_o} = 0; \frac{\partial \rho}{\partial s}_{x_o y_o z_o} = 0$$

Non-uniform flow: Flow in which the fluid characteristics like velocity, pressure, density etc at a point changes with "space".

$$\frac{\partial v}{\partial s}_{x_o y_o z_o} \neq 0; \frac{\partial p}{\partial s}_{x_o y_o z_o} \neq 0; \frac{\partial \rho}{\partial s}_{x_o y_o z_o} \neq 0$$

3. ***Laminar and Turbulent flows:***

Laminar flow: the fluid particle move along well defined path or streamline and all the streamlines are straight and parallel. Thus the particles moves in laminar or layers gliding smoothly over the adjacent layer. Also called stream line flow/viscous flow.

Turbulent flow: the fluid particles moves in a zig-zag way due to this high energy loss causes due to creation of eddies.

The type of flow in pipes are determined by Reynold's non-dimensional number Re = $\frac{VD}{\nu}$

If Re < 2000 ----- Laminar flow

If Re > 4000 ----- Turbulent flow

If Re >2000 and < 4000 --- Transition flow

4. ***Compressible and Incompressible flows:***

Compressible flow: Density of the fluid changes from point to point (or) in other words the density is not constant for the liquid.

$\rho \neq$ constant

Incompressible flows: Density of the fluid doesn't changes from point to point (or) in other words the density is constant for the liquid.

$\rho =$ constant

5. ***Rotational and Irrotational Flow:***

A particle which flow along the streamlines and also rotates about their own axis is called rotational flow. if the fluid particles while flowing along streamlines and donot rotates about their own axis then that type of flow is irrotational flow.

6. ***One, Two and Three Dimensional Flows:***

When the flow and fluid properties are function of time and one space co-ordinate (x-axis) only then the flow is said to be one-dimensional flow. Similarly if the flow and fluid properties are function of time and two (x & Y axis) or three (x, y & z) space co-ordinate only then the flow is said to be two or three -dimensional flow respectively.

RATE OF FLOW OR DISCHARGE:

"The quantity of a fluid flowing per second through a known section in pipe or channel."

Representation: **Q**

Units: **m^3/sec** for liquids

kgf/sec for gases

CONTINUITY EQUATION:

The equation developed based on the principle of conservation of mass is called “continuity equation”

Thus for a fluid flowing through a pipe at all the cross sections the quantity of fluid per second is constant.

Consider two c/s’s of a pipe as shown in fig.

Let V_1 = avrg. Velocity at section 1-1

ρ_1 = Density at section C/s 1-1

A_1 = Area of the pipe at section 1-1

and V_2,ρ_2 and A_2 are corresponding values at section 2-2

According to the law of conservation of mass,

Rate of flow at section 1-1 = Rate of flow at section 2-2

$\boldsymbol{\rho_1 A_1 V_1 = \rho_2 A_2 V_2}$

Eq(1) is applicable for the compressible as well as incompressible fluids is called “continuity equation”.

If the fluid is incompressible, then $\rho_1 = \rho_2$ and the Continuity equation reduces to

$$\mathbf{A_1 V_1 = A_2 V_2}$$

PROBLEMS- CONTINUITY EQUATION

1. The dia. Of a pipe at section 1 and section 2 are 10cm and 15cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 is 5m/s. determine the velocity at section 2.

Ans. 2.22 m/s

Fig

2. A 30cm dia. pipe, conveying water, branches into two pipes of dia. 20cm and 15cm respectively. If the velocity in 30cm dia. pipe is 2.5m/s. find the discharge in this pipe. Also determine the V in 15cm pipe. If the Vavrg. In 20cm dia. pipe is 2m/sec.

Ans. Q1 = 0.1767 m3/sec, V3 = 6.44 m/sec

Fig.

3. Water flows through a pipe AB 1.2m dia at 3 m/s and then passes through a pipe BC 1.5m dia. At C, the pipe branches. Branch CD is 0.8m in dia. and carries one third of the flow in AB. The flow velocity in branch CE is 2.5 m/s. fid the volume rate of flow in AB, the velocity in BC, the velocity in CD and the dia. of CE.

Ans. Diameter of Pipe CE = 1.0735 m.

Fig.

4. A jet of water from a 25mm dia. nozzle is directed vertically upwards. Assuming that the jet remains circular and neglecting any loss of energy, that will be the dia. @ a point 4.5m above the nozzle, if the velocity with which the jet leaves the nozzles is 12 m/sec.
   Ans. D2 = 31.7 mm

   Fig.

CONTINUITY EQUATION IN TWO DIMENSIONS

Consider a fluid element of length dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively.

Mass of fluid entering the face ABCD per second = ρ*Velocity in x-direction * Area of ABCD

= ρ * u * (dy*dz)

Then mass of fluid leaving the face EFGH per second = ρ * u * (dy*dz) + $\frac{\partial}{\partial x}(\rho u\ dydz)dx$

∴ Gain of mass in x-direction = (Mass through ABCD - Mass through EFGH) per second

= ρ * u * (dy*dz) - ρ * u * (dy*dz) - $\frac{\partial}{\partial x}(\rho u\ dydz)dx$

= - $\frac{\partial}{\partial x}(\rho u)\ dxdydz$

Similarly, the net gain of mass in y-direction

= - $\frac{\partial}{\partial y}(\rho v)\ dxdydz$

and in z-direction = - $\frac{\partial}{\partial z}(\rho w)\ dxdydz$

∴ Net gain of masses = - $\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho * dx * dy * dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho dxdydz)$ or $\frac{\partial \rho}{\partial t} * dx\ dy\ dz.$

Equating the two expressions,

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t} * dx\, dy\, dz.$$

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] = 0 \qquad \text{------- (1)}$$

Equation (1) is the Continuity equation in Cartesian co-ordinates in its most general form.

This equation is applicable to:

(i) Steady and unsteady flow,

(ii) Uniform and non-uniform flow and

(iii) Compressible and non-compressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t}$ = 0 and hence equation (1) becomes as

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] = 0 \qquad \text{-------- (2)}$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \qquad \text{-------- (3)}$$

Equation (3) is the continuity equation in three-dimensions.

For a two-dimensional flow, the component **w** = 0 and hence the continuity equation reduces to

$$\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0 \qquad \text{--------- (4)}$$

VELOCITY AND ACCELERATION

Let V us the resultant velocity at any point in a fluid flow. Let u, v and w are its components in x, y and z directions. The velocity components are functions of space-co-ordinates and time.

Mathematically the velocity components are given as

$$u = f_1 (x, y, z, t)$$
$$v = f_2 (x, y, z, t)$$
$$w = f_3 (x, y, z, t)$$

and Resultant velocity $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y, and a_z are the total acceleration in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w$

∴

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t} \\ a_y &= \frac{dv}{dt} = \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w + \frac{\partial v}{\partial t} \\ a_z &= \frac{dw}{dt} = \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{\partial w}{\partial t} \end{aligned}\right] \qquad \text{-------- (1)}$$

For steady flow, $\frac{\partial V}{\partial t}$ = 0, where V is resultant velocity

Or $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$

Hence acceleration in x, y and z directions becomes

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w \\ a_y &= \frac{dv}{dt} = \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w \\ a_z &= \frac{dw}{dt} = \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w \end{aligned}\right] \quad \text{-------- (2)}$$

Acceleration vector

$$\left.\begin{aligned} A &= a_x i + a_y j + a_z k \\ &= \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2} \end{aligned}\right] \quad \text{--------- (3)}$$

- **Local Acceleration** is defined as the rate of increase of velocity with respect to time at a given point in a flow filed. In the equation given by (1) , the expression $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ is known as local acceleration.
- **Convective acceleration** is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expression other than $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ is known as convective acceleration.

PROBLEMS ON VELOCITY AND ACCELERATION OF FLUID PARTICLE

1. **The velocity vector in a fluid flow is given V = $4x^3i - 10x^2yj + 2tk$. Fins the velocity and acceleration of fluid particle at (2,1,3) at time t = 1.**

Sol. The velocity components u,v and w are $u = 4x^3$, $v = -10x^2y$, $w = 2t$

For the point (2,1,3), we have x=2, y=1 and z = 3 at time t = 1

Hence velocity components at (2,1,3) are

$$u = 4*(2)^3 = 32 \text{ units}$$
$$v = -10(2)^2*1 = -40 \text{ units}$$
$$w = 2*1 = 2 \text{ units}$$

∴ Velocity vector V at (2,1,3) = 32i – 40j + 2k

Resultant velocity = $\sqrt{u^2+v^2+w^2} = \sqrt{32^2+(-40)^2+2^2}$ = 51.26 units

Acceleration is given by equations

$$a_x = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w$$
$$a_y = \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w$$
$$a_z = \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w$$

Now from velocity components, we have

$\frac{\partial u}{\partial x} = 12x^2$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial u}{\partial z} = 0$, $\frac{\partial u}{\partial t} = 0$

$\frac{\partial v}{\partial x} = -20\,xy$, $\frac{\partial v}{\partial y} = -10x^2$, $\frac{\partial v}{\partial z} = 0$, $\frac{\partial v}{\partial t} = 0$

$\frac{\partial w}{\partial x} = 0$, $\frac{\partial w}{\partial y} = 0$, $\frac{\partial w}{\partial z} = 0$ and $\frac{\partial w}{\partial t} = 2.1$

Substituting the values, the acceleration components at (2,1,3) at time t = 1 are

$a_x = 4x^3(12x^2) + (-10x^2y)(0) + 2t*0 + 0 = 48x^5 = 48*(2)^5 = 48*32 = 1536$ units

$a_y = 4x^3(-20xy) + (-10x^2y)(-10x^2) + 2t*0 + 0$

$= -80x^4y + 100\,x^4y = -80(2)^4(1) + 100\,(2)^4(1) = -1280+1600 = 320$ units

$a_z = 4x^3(0) + (-10x^2y)(0) + 2t*0 + 2.1 = 2$ units

∴ Acceleration is $A = a_xi + a_yj + a_zk = 1536i + 320j + 2k$

∴ Resultant = $\sqrt{1536^2+320^2+2^2}$ = 1568.9 units

2. **A fluid flow field is given by V = $x^2yi + y^2zj - (2xyz + yz^2)k$ prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point (2,1,3)**

Sol: For the given fluid flow filed $u = x^2y$ ∴ $\frac{\partial u}{\partial x} = 2xy$

$v = y^2z$ ∴ $\frac{\partial v}{\partial y} = 2yz$

$w = -2xyz - yz^2$ ∴ $\frac{\partial w}{\partial z} = -2xy - 2yz$

For a case of possible steady incompressible fluid flow, the continuity equation should be satisfied.

i.e., $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

Substituting the values of $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$, we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 2xy + 2yz - 2xy - 2yz = 0$$

Hence the velocity filed $V = x^2y\,i + y^2z\,j - (2xyz + yz^2)k$ is apossible case of fluid flow.

Velocity at (2,1,3)

Substituting the values x =2, y =1 and z = 3 in velocity filed, we get

$$V = x^2y\,i + y^2z\,j - (2xyz + yz^2)k$$
$$= 2^2*1i + 1^2*3j - (2*2*1*3 + 1*3^2)k$$
$$= 4i + 3j - 21k.$$

Resultant velocity = $\sqrt{4^2 + 3^2 + (-21)^2}$ = 21.587 units

Acceleration at (2,1,3)

Acceleration is given by equations

$$a_x = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w$$
$$a_y = \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w$$
$$a_z = \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w$$

$u= x^2y$, $\frac{\partial u}{\partial x} = 2xy$, $\frac{\partial u}{\partial y} = x^2$, $\frac{\partial u}{\partial z} = 0$

$v= y^2z$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 2yz$, $\frac{\partial v}{\partial z} = y^2$

$w = -2xyz - yz^2$, $\frac{\partial w}{\partial x} = -2yz$, $\frac{\partial w}{\partial y} = -2xz-z^2$, $\frac{\partial w}{\partial z} = -2xy-2yz$

Substituting these values in acceleration components, we get acceleration at (2,1,3)

$ax = x^2y\,(2xy) + y^2z\,(x^2) - (2xyz + yz^2)\,(y^2) = 28$ units

ay = -3 units

az = 123 units

∴ Acceleration = **28i – 3j + 123k**

VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

Velocity potential:

It is defined as a scalar function of space and time such that it's negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi).

Mathematically, the velocity potential is defined as ϕ = f (x, y, z) for steady flow such that

$u = \frac{-\partial\varphi}{\partial x}$; $v = \frac{-\partial\varphi}{\partial y}$; $w = \frac{-\partial\varphi}{\partial z}$

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Substituting the values of u, v and w from above equation, we get

$$\frac{\partial}{\partial x}\left(\frac{-\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{-\partial\varphi}{\partial z}\right) = 0$$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

The above equation is a laplace equation

For two-dimension case, the equation reduces to $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$

Properties of the potential function

The rotational components (the movement of the fluid element in such a way that both of its axes (horizontal as well as vertical) rotate in the same direction)

For element in x-y plane, $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

For element in y-z plane, $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

For element in z-x plane, $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

Substituting the components of u, v, w in rotational components, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{-\partial\varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{-\partial\varphi}{\partial x}\right)\right] = \frac{1}{2}\left[\left(\frac{-\partial^2\varphi}{\partial x\partial y}\right) + \left(\frac{\partial^2\varphi}{\partial y\partial x}\right)\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(\frac{-\partial\varphi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{-\partial\varphi}{\partial z}\right)\right] = \frac{1}{2}\left[\left(\frac{-\partial^2\varphi}{\partial z\partial x}\right) + \left(\frac{\partial^2\varphi}{\partial x\partial z}\right)\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{-\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{-\partial\varphi}{\partial y}\right)\right] = \frac{1}{2}\left[\left(\frac{-\partial^2\varphi}{\partial y\partial z}\right) + \left(\frac{\partial^2\varphi}{\partial z\partial y}\right)\right]$$

If ϕ is a continuous function then $\frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x}$; $\frac{\partial^2\varphi}{\partial z\partial x} = \frac{\partial^2\varphi}{\partial x\partial z}$; $\frac{\partial^2\varphi}{\partial y\partial z} = \frac{\partial^2\varphi}{\partial z\partial y}$

$\therefore \omega_z = \omega_y = \omega_x$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are:

1. If the velocity potential exists, the flow should be irrotational.
2. If the velocity potential satisfies the Laplace equation, it represents the possible case of the steady incompressible irrotational flow.

Stream Function:

It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity components at the right angles to that direction. It is denoted by ψ (Psi) and defined only for two-dimensional flow.

Mathematically, for steady flow it is defined as ψ = f (x, y) such that

$\frac{\partial\psi}{\partial x} = v$; $\frac{\partial\psi}{\partial y} = -u$

Properties of stream function

The continuity equation for two – dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Substituting the values of u and v from above equation, we get

$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$

$\left(\frac{\partial^2 \psi}{\partial x \partial y}\right) + \left(\frac{\partial^2 \psi}{\partial y \partial x}\right) = 0$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

Substituting the values of u and v in rotational components, we get

$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left[\left(\frac{\partial^2 \psi}{\partial x^2}\right) + \left(\frac{\partial^2 \psi}{\partial y^2}\right)\right]$

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\left(\frac{\partial^2 \psi}{\partial x^2}\right) + \left(\frac{\partial^2 \psi}{\partial y^2}\right) = 0$

Hence the properties of stream function are:

1. If the stream function exists, it is a possible case of fluid flow which may be rotational or irrotational.
2. If stream function satisfies the Laplace equation, it is possible case of an irrotational flow.

EQUIPOTENTIAL LINE:

A line along which the velocity potential ϕ is constant, is called equipotential line.

i.e., ϕ = constant

FLOW NET:

A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analyzing two-dimensional irrotational flow problems.

Relation Between stream function and velocity potential function

We know that,

$u = \frac{-\partial \varphi}{\partial x}$; $v = \frac{-\partial \varphi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$; $u = -\frac{\partial \psi}{\partial y}$

Thus, we have $u = \frac{-\partial \varphi}{\partial x} = -\frac{\partial \psi}{\partial y}$ and $v = \frac{-\partial \varphi}{\partial y} = \frac{\partial \psi}{\partial x}$

Hence $\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$

STREAM LINES, STREAM TUBE, PATH LINES, STREAK LINES AND TIME LINES

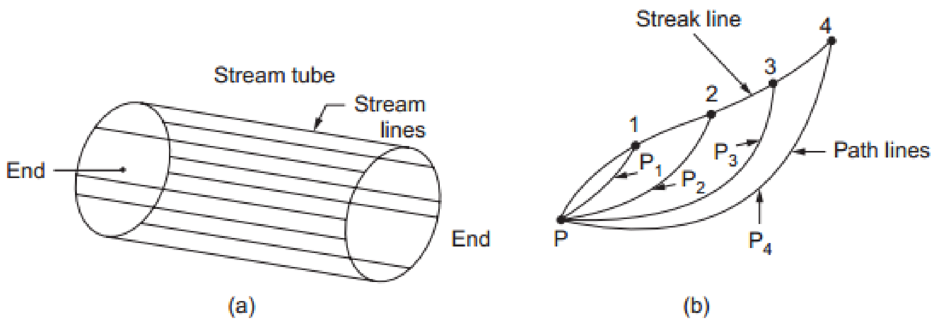
The analytical description of flow velocity is geometrically depicted through the concept of stream lines. The velocity vector is a function of both position and time. If at a fixed instant of time a curve is drawn so that it is tangent everywhere to the velocity vectors at these locations then the curve is called a stream line. Thus stream line shows the mean direction of a number of particles in the flow at the same instant of time. **Stream lines are a series of curves drawn tangent to the mean velocity vectors of a number of particles in the flow. Since stream lines are tangent to the velocity vector at every point in the flow field, there can be no flow across a stream line.**

A bundle of neighbouring stream lines may be imagined to form a passage through which the fluid flows. Such a passage is called a stream tube. Since the stream tube is bounded on all sides by stream lines, there can be no flow across the surface. Flow can be only through the ends.

A stream tube is shown diagrammatically in Figure. Under steady flow condition, the flow through a stream tube will be constant along the length.

Path line is the trace of the path of a single particle over a period of time. Path line shows the direction of the velocity of a particle at successive instants of time. In steady flow path lines and stream lines will be identical.

Streak lines provide an instantaneous picture of the particles, which have passed through a given point like the injection point of a dye in a flow. In steady flow these lines will also coincide with stream lines. Path lines and streak lines are shown in Figure



Particles P_1, P_2, P_3, P_4, starting from point P at successive times pass along path lines shown. At the instant of time considered the positions of the particles are at 1, 2, 3 and 4. A line joining these points is the streak line.

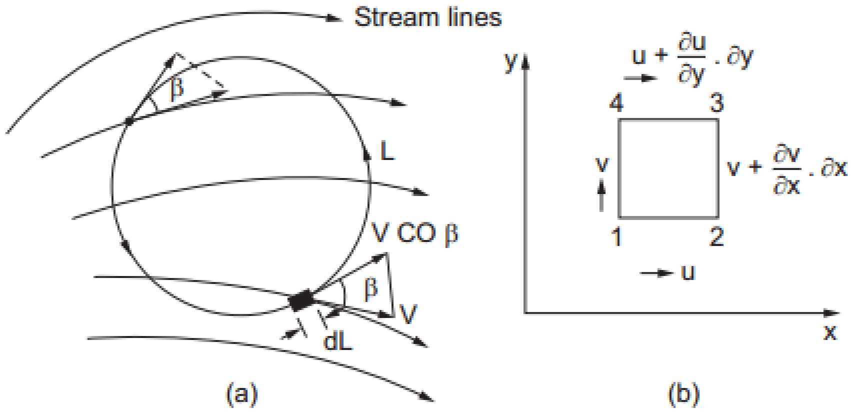
If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line at that instant. This line is called time line. Subsequent observations of the line may provide information about the flow field. For example the deformation of a fluid under shear force can be studied using time lines.

CONCEPTS OF CIRCULATION AND VORTICITY

Considering a closed path in a flow field as shown in Fig. , circulation is defined as the line integral of velocity about this closed path. The symbol used is Γ.

$$\Gamma = \oint_L u \; ds = \oint_L u \; \cos\beta \; dL$$

where dL is the length on the closed curve, u is the velocity at the location and β is the angle between the velocity vector and the length dL. The closed path may cut across several stream lines and at each point the direction of the velocity is obtained from the stream line, as its tangent at that point.



The integration can be performed over an element as shown in Fig.

In the cartesian co-ordinate if an element *dx*. *dy* is considered, then the circulation can be calculated as detailed below:
Consider the element 1234 in Fig., Starting at 1 and proceeding counter clockwise,

$$d\Gamma = u\,dx + [v+(\partial v/\partial x)dx]dy - [u+(\partial u/\partial y).dy]\,dx - vdy$$
$$= [\partial v/\partial x - \partial u/\partial y]dxdy$$

Vorticity is defined as circulation per unit area. i.e., Vorticity = circulation per unit area, here area is *dx dy*, so

$$\text{Vorticity} = \frac{d\Gamma}{dxdy} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

For irrotational flow, vorticity and circulation are both zero.

PROBLEMS

1. **The velocity potential function is given by ϕ = 5 (x^2-y^2). Calculate the velocity components at the point (4,5).**

Sol. : Given ϕ = 5 (x^2-y^2)

$$\frac{\partial \varphi}{\partial x} = 10x$$

$$\frac{\partial \varphi}{\partial y} = -10y$$

But velocity components u and v are given by equation

$u = -\frac{\partial \varphi}{\partial x} = -10x$

$v = -\frac{\partial \varphi}{\partial y} = -(-10y) = 10y$

The velocity components at the point (4,5) i.e., at x = 4, y = 5

u = -10*4 = -40 units

v = 10*5 = 50 units

2. **If for a two-dimensional potential flow, the velocity potential is given by ϕ = x (2y - 1). Determine the velocity at the point P (4, 5). Determine also the value of stream function ψ at the point P.**

Sol. :

(*i*) The velocity components in the direction of *x* and *y* are

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}[x(2y-1)] = -[2y-1] = 1-2y$$

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}[x(2y-1)] = -[2x] = -2x$$

At the point *P* (4, 5), *i.e.*, at *x* = 4, *y* = 5

$u = 1 - 2 \times 5 = -9$ units/sec

$v = -2 \times 4 = -8$ units/sec

∴ Velocity at *P* $= -9i - 8j$

or Resultant velocity at *P* $= \sqrt{9^2 + 8^2} = \sqrt{81+64}$ = 12.04 units/sec = **12.04 units/sec.** A

(*ii*) **Value of Stream Function at P**

We know that $\frac{\partial \psi}{\partial y} = -u = -(1-2y) = 2y - 1$

and $\frac{\partial \psi}{\partial x} = v = -2x$

Integrating equation (*i*) w.r.t. 'y', we get

$$\int d\psi = \int (2y-1)\,dy \text{ or } \psi = \frac{2y^2}{2} - y + \text{Constant of integration.}$$

The constant of integration is not a function of y but it can be a function of x. Let the value of constant of integration is k. Then

$$\psi = y^2 - y + k. \qquad ...(iii)$$

Differentiating the above equation w.r.t. 'x', we get

$$\frac{\partial \psi}{\partial x} = \frac{\partial k}{\partial x}.$$

But from equation (*ii*), $\frac{\partial \psi}{\partial x} = -2x$

Equating the value of $\frac{\partial \psi}{\partial x}$, we get $\frac{\partial k}{\partial x} = -2x$.

Integrating this equation, we get $k = \int -2x dx = -\frac{2x^2}{2} = -x^2$.

Substituting this value of k in equation (*iii*), we get $\psi = \mathbf{y^2 - y - x^2}$. **Ans.**

∴ Stream function ψ at $P(4, 5) = 5^2 - 5 - 4^2 = 25 - 5 - 16 = \mathbf{4}$ **units. Ans.**

3. **The stream function for two-dimensional flow is given by ψ = 2xy, calculate the velocity at the point P (2,3). Find the velocity potential function ϕ.**

Sol. :

The velocity components u and v in terms of ψ are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(2xy) = -2x$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y.$$

At the point P (2, 3), we get $u = -2 \times 2 = -4$ units/sec

$v = 2 \times 3 = 6$ units/sec

∴ Resultant velocity at $P = \sqrt{u^2 + v^2} = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52} = 7.21$ units/sec.

Velocity Potential Function ϕ

We know $\frac{\partial \phi}{\partial x} = -u = -(-2x) = 2x$

$$\frac{\partial \phi}{\partial y} = -v = -2y$$

Integrating equation (*i*), we get

$$\int d\phi = \int 2x dx$$

or

$$\phi = \frac{2x^2}{2} + C = x^2 + C$$

where C is a constant which is independent of x but can be a function of y.

Differentiating equation (*iii*) w.r.t. 'y', we get $\frac{\partial \phi}{\partial y} = \frac{\partial C}{\partial y}$

But from (*ii*), $\frac{\partial \phi}{\partial y} = -2y$

$\therefore$ $\frac{\partial C}{\partial y} = -2y$

Integrating this equation, we get $C = \int -2y\,dy = -\frac{2y^2}{2} = -y^2$

Substituting this value of C in equation (*iii*), we get $\phi = \mathbf{x^2 - y^2}$. **Ans.**

4. **In a two – dimensional incompressible flow, the fluid velocity components are given by u = x – 4y and v = -y-4x. Show that velocity potential exists and determine its form. Find also the stream function.**

Sol. : Given, **u = x – 4y and v = -y-4x**

$$\therefore \quad \frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

Hence flow is continuous and velocity potential exists.

Let ϕ = Velocity potential.

Let the velocity components in terms of velocity potential is given by

$$\frac{\partial \phi}{\partial x} = -u = -(x - 4y) = -x + 4y \quad \ldots(i)$$

and

$$\frac{\partial \phi}{\partial y} = -v = -(-y - 4x) = y + 4x \quad \ldots(ii)$$

Integrating equation (*i*), we get $\phi = -\frac{x^2}{2} + 4xy + C$ ...(*iii*)

where C is a constant of integration, which is independent of x.

This constant can be a function of y.

Differentiating the above equation, *i.e.*, equation (*iii*) with respect to 'y', we get

$$\frac{\partial \phi}{\partial y} = 0 + 4x + \frac{\partial C}{\partial y}$$

But from equation (*iii*), we have $\frac{\partial \phi}{\partial y} = y + 4x$

Equating the two values of $\frac{\partial \phi}{\partial y}$, we get

$$4x + \frac{\partial C}{\partial y} = y + 4x \quad \text{or} \quad \frac{\partial C}{\partial y} = y$$

Integrating the above equation, we get

$$C = \frac{y^2}{2} + C_1$$

where C_1 is a constant of integration, which is independent of x and y.

Taking it equal to zero, we get $C = \frac{y^2}{2}$.

Substituting the value of C in equation (*iii*), we get

$$\phi = -\frac{x^2}{2} + 4xy + \frac{y^2}{2}. \textbf{ Ans.}$$

Value of Stream functions

Let ψ = Stream function

The velocity components in terms of stream function are

$$\frac{\partial \psi}{\partial x} = v = -y - 4x \quad \text{...(iv)}$$

and

$$\frac{\partial \psi}{\partial y} = -u = -(x - 4y) = -x + 4y \quad \text{...(v)}$$

Integrating equation (*iv*) w.r.t. x, we get

$$\psi = -yx - \frac{4x^2}{2} + k \quad \text{...(vi)}$$

where k is a constant of integration which is independent of x but can be a function of y.

Differentiating equation (*vi*) w.r.t. y, we get $\frac{\partial \psi}{\partial y} = -x - 0 + \frac{\partial k}{\partial y}$

But from equation (v), we have $\frac{\partial \psi}{\partial y} = -x + 4y$

Equating the two values of $\frac{\partial \psi}{\partial y}$, we get $-x + \frac{\partial k}{\partial y} = -x + 4y$ or $\frac{\partial k}{\partial y} = 4y$

Integrating the above equation, we get $k = \frac{4y^2}{2} = 2y^2$

Substituting the value of k in equation (*vi*), we get

$$\psi = -yx - 2x^2 + 2y^2. \textbf{ Ans.}$$