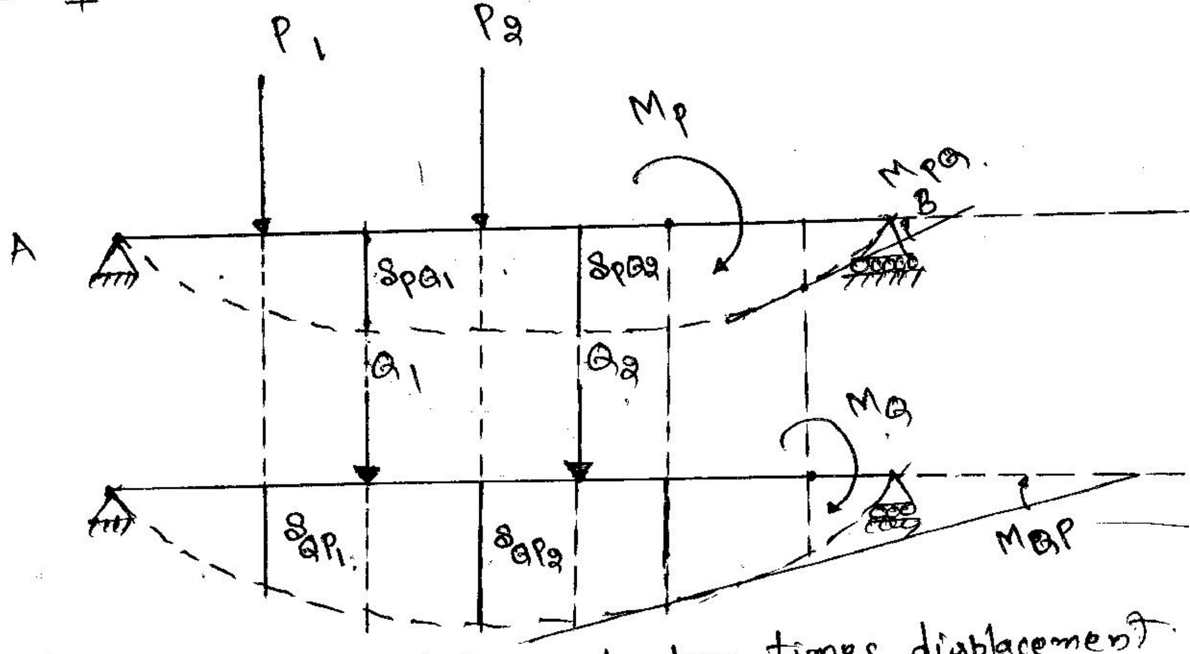
Energy theorems

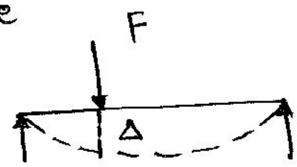
1. Betti's theorem:

In a linear elastic structure, The sum of individual Workdónes by the load system 1 produced by individual loads in the loadsystem 1 and deflections caused by load system 2 under position of loads in load system 1 equal to the sum of individual work dones by the load system 2 produced by individual loads in the load system 2 & relevant deflections caused by loads in system 2 under the position of loads in loadsystem 1.

~~loads~~



Here, work done = Force on a body or structure times displacement of that body or structure



$= F \times \Delta$

& also

Work done = Moment acting on a body or structure times rotation of that body or structure

$= M \times \theta$

$$\boxed{P_1 \delta_{QP_1} + P_2 \delta_{QP_2} + M_P \theta_{QP} = Q_1 \delta_{PQ_1} + Q_2 \delta_{PQ_2} + M_Q \theta_{PQ}}$$

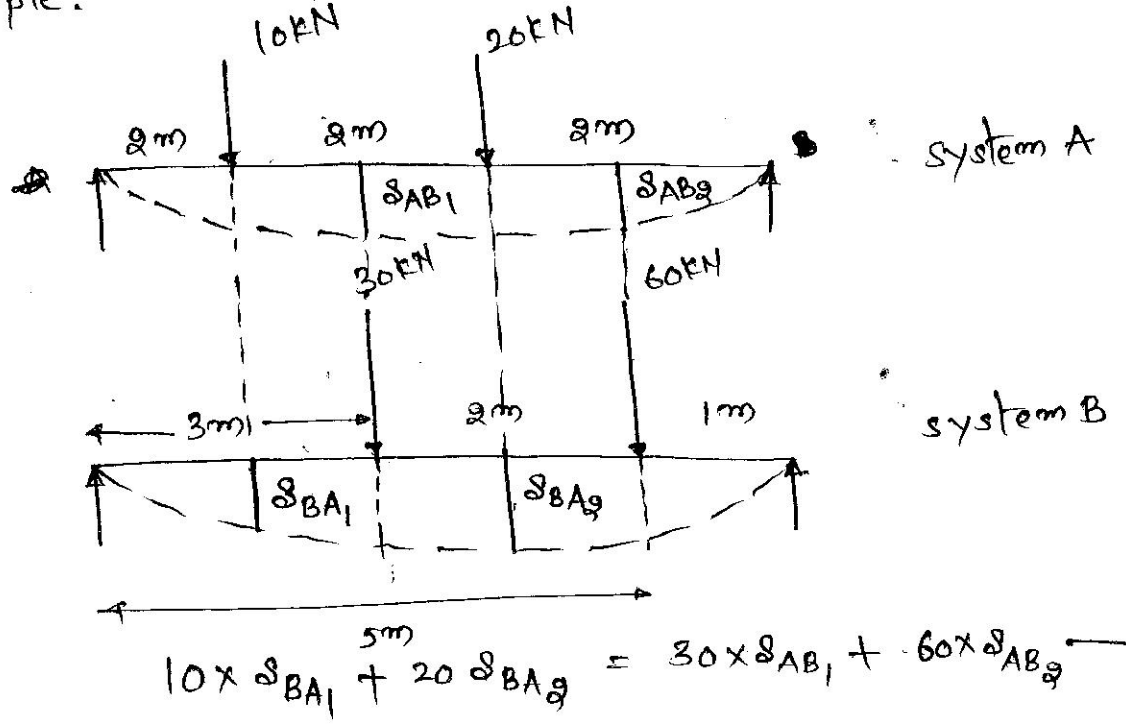
δ_{QP_1} = Deflection caused by load system 2 under position of load P_1

δ_{QP_2} = Deflection caused by load system 2 under position of load P_2

$M_P\, \theta_{QP}$ = slope/rotation caused by load system 2 under position of moment 'M_P'.

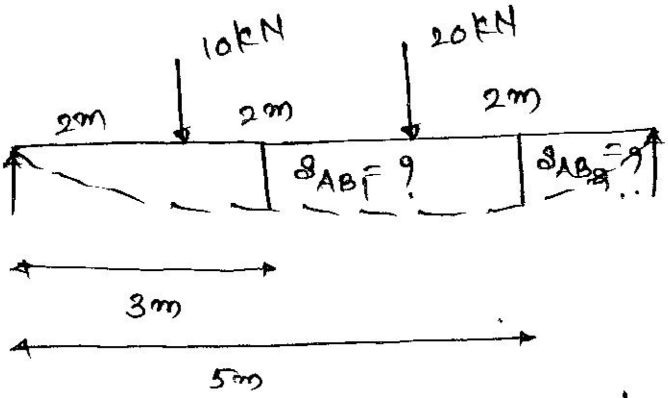
Similarly δ_{PQ_1}, δ_{PQ_2} & θ_{PQ}

Betti's theorem can be proved logically using a numerical example.

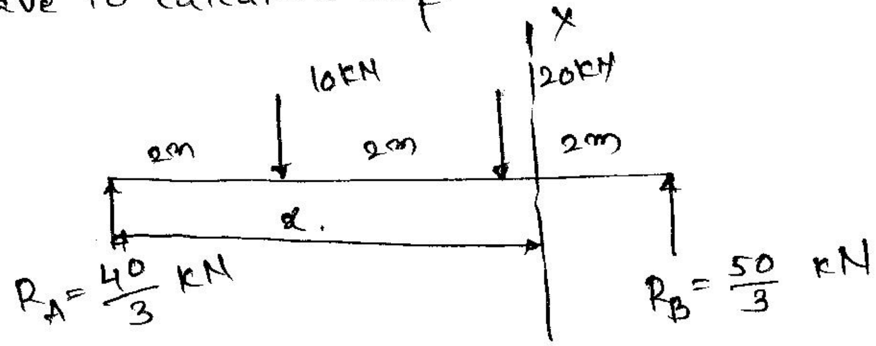


$$10 \times \delta_{BA_1} + 20\, \delta_{BA_2} = 30 \times \delta_{AB_1} + 60 \times \delta_{AB_2} \quad \text{— (A)}$$

First, we have to calculate the four deflections above using conventional methods such as Double integration method - Macaulay's method, Moment area method & Conjugate beam method start from system A



We have to calculate deflections at 3m & 5m



Using Double integration method

$$M_x = R_A x - 10(x-2) - 20(x-4)$$

$$EI\frac{d^2y}{dx^2} = -M_x. \qquad R_A = \frac{40}{3}\text{kN}$$

$$EI\frac{d^2y}{dx^2} = -\frac{40}{3}x + 10(x-2) + 20(x-4)$$

Integrating

$$EI\frac{dy}{dx} = -\frac{40}{3}\cdot\frac{x^2}{2} + \frac{10(x-2)^2}{2} + 20\frac{(x-4)^2}{2} + C_1$$

Again integrating

$$EIy = \frac{-40}{6}\times\frac{x^3}{3} + \frac{10}{2}\cdot\frac{(x-2)^3}{3} + \frac{20}{2}\frac{(x-4)^3}{3} + C_1x + C_2$$

Applying boundary conditions

put $x=0$; $y=0$ $\quad C_2 = 0$

put $x = 6m$ $\quad y=0$

$$0 = \frac{-40}{3\times 6}(6)^3 + \frac{10}{6}(4)^3 + \frac{20}{6}(2)^3 + C_1(6)$$

$$0 = \frac{-1040}{3} + 6C_1 \Rightarrow \boxed{C_1 = \frac{520}{9}}$$

Deflection at 3m $y = \delta_{AB_1}$

$$\boxed{EIy = -\frac{40}{6}\frac{x^3}{3} + \frac{10}{6}(x-2)^3 + \frac{20}{6}(x-4)^3 + \frac{520}{9}x}$$

$$EI\,\delta_{AB_1} = -\frac{40}{18}(3)^3 + \frac{10}{6}(1)^3 + \frac{520}{9}(3)$$

(-ve terms neglected)

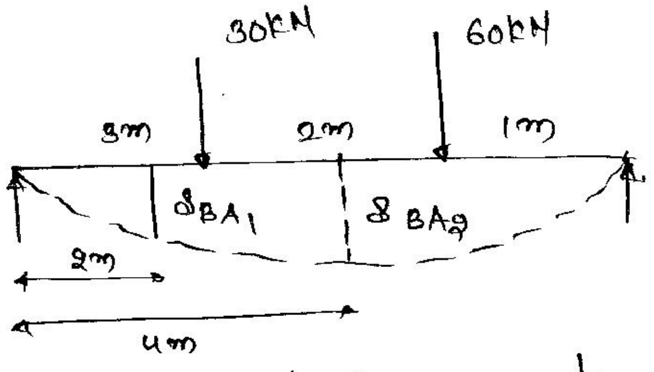
$$\boxed{\delta_{AB_1} = \frac{115}{EI}} \text{ — I}$$

Deflection at $x=5m$; $y = \delta_{AB_2}$

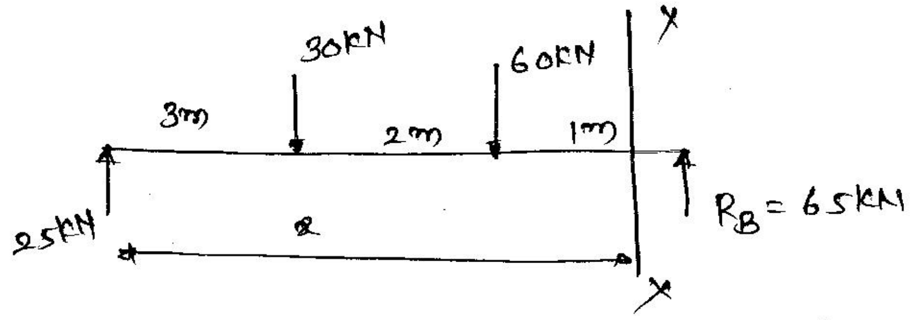
$$EI\,\delta_{AB_2} = -\frac{40}{18}(5)^3 + \frac{10}{6}(3)^3 + \frac{20}{6}(1)^3 + \frac{520}{9}(5)$$

$$\boxed{\delta_{AB_2} = \frac{535}{9EI}} \text{ — II}$$

System B:



We have to calculate deflections at 2m & 4m respectively



$$M_x = 25x - 30(x-3) - 60(x-5)$$

$$EI\frac{d^2y}{dx^2} = -M_x$$

$$\Rightarrow EI\frac{d^2y}{dx^2} = -25x + 30(x-3) + 60(x-5)$$

Integrating

$$EI\frac{dy}{dx} = -\frac{25x^2}{2} + \frac{30(x-3)^2}{2} + \frac{60(x-5)^2}{2} + C_1 \quad -(1)$$

Again Integrating

$$EIy = -\frac{25}{2}\frac{x^3}{3} + \frac{30}{2}\frac{(x-3)^3}{3} + \frac{60}{2}\frac{(x-5)^3}{3} + C_1x + C_2 \quad -(2)$$

put $x = 0$; $y = 0$; $C_2 = 0$ neglecting (-ve power terms

put $x = 6$; $y = 0$ in eq(2)

$$0 = -\frac{25}{6}\times 6^3 + \frac{30}{6}(3)^3 + \frac{60}{6}(1)^3 + 6C_1$$

$$\Rightarrow \quad 0 = -755 + C_1 6$$

$$\boxed{C_1 = \frac{755}{6}}$$

Therefore,

$$EIy = -\frac{25}{6}x^3 + \frac{30}{6}(x-3)^3 + \frac{60}{6}(x-5)^3 + \frac{755x}{6}$$

put $x = 2m$; $y = \delta_{BA_1}$

$$EI\delta_{BA_1} = -\frac{25}{6}(2)^3 + \frac{755}{6}(2)$$ (-ve power terms neglected

$$\boxed{\delta_{BA_1} = \frac{655}{3EI}} \quad - \text{III}$$

put $x = 4m$

$$EI\,\delta_{BA_2} = -\frac{25}{6}(4)^3 + \frac{30}{6}(1)^3 + \frac{755}{6}(4)$$

neglecting (-ve) power terms

$$\boxed{\delta_{BA_2} = \frac{725}{3EI}} \quad \text{— } \overline{\text{IV}}$$

From Betti's theorem

eq (A) $\Rightarrow$ $10 \times \delta_{BA_1} + 20\,\delta_{BA_2} = 30\,\delta_{AB_1} + 60 \times \delta_{AB_2}$

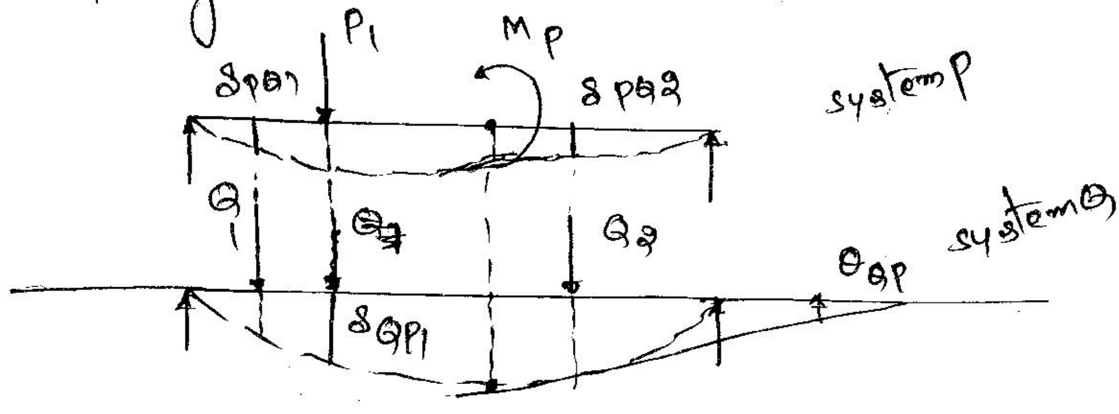
$$10 \times \frac{655}{3EI} + 20 \times \frac{725}{3EI} \;;\; 30 \times \frac{115}{EI} + \frac{60 \times 535}{9EI}$$

$$\frac{21050}{3EI} \;;\; \frac{21050}{3}$$

Hence, it is proved. (logically)

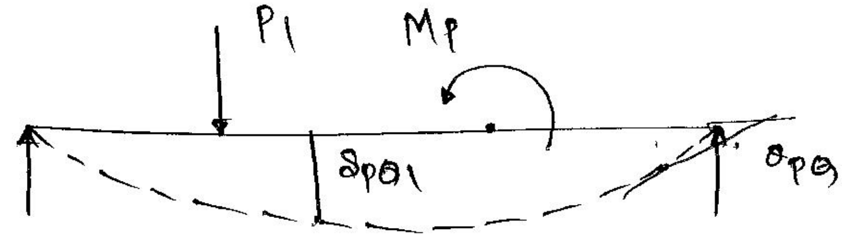
Typical cases:

Case(i): Applying moment in anticlockwise direction in one system

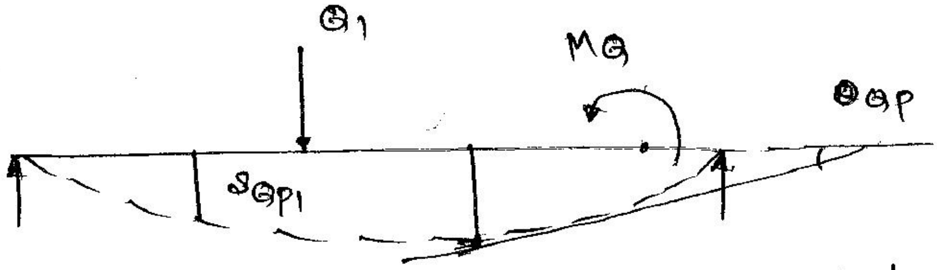


$$P_1\,\delta_{QP_1} + M_P(-\theta_{QP}) = Q_1\,\delta_{PQ_1} + Q_2\,\delta_{PQ_2}$$

Case(ii): 'M' anticlockwise direction in both systems



system P



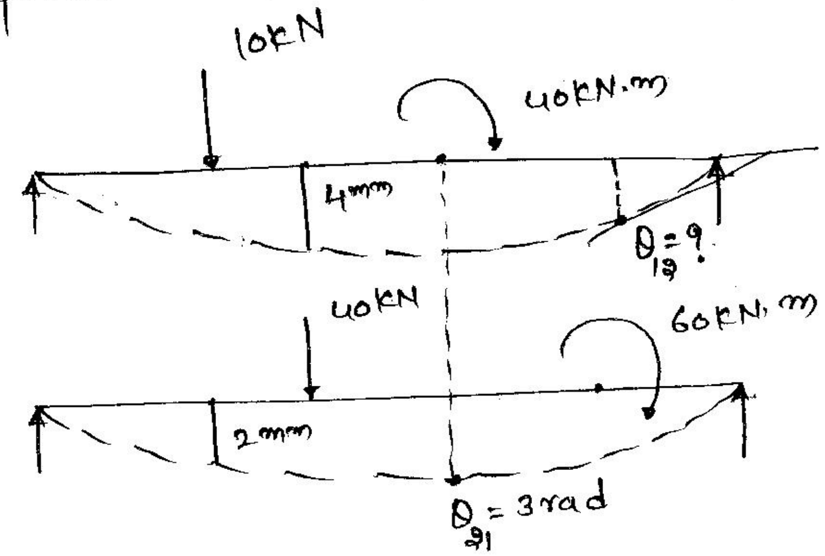
system Q

$$P_1 \delta_{QP_1} + M_P(-\theta_{QP}) = Q_1 \delta_{PQ_1} + M_Q(-\theta_{PQ})$$

Actually anticlockwise moment containg upward rotation takes (-)ve.

Exercize problems

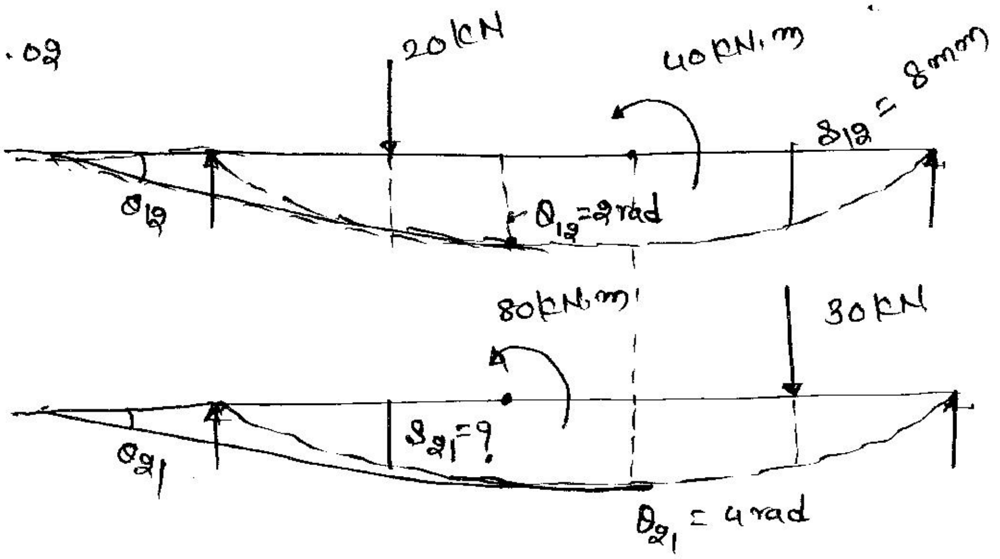
Ex. 01



system 1

system 2

Ex. 02



system 1

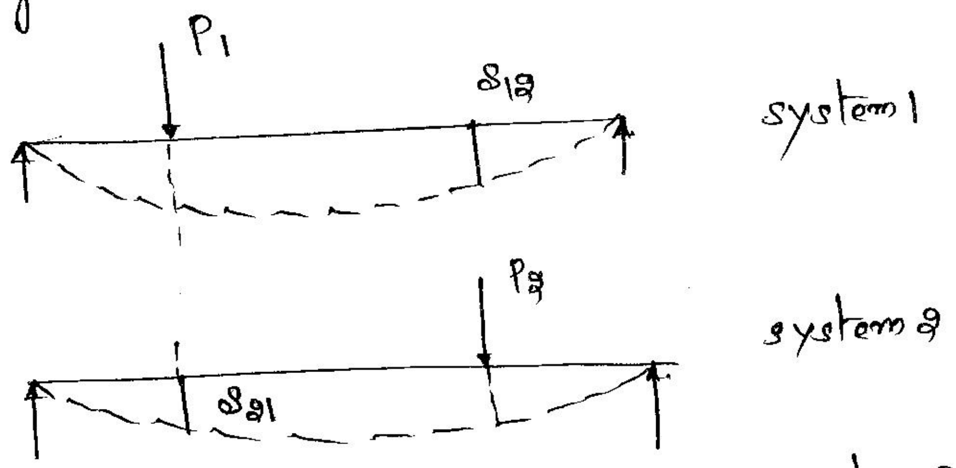
system 2

Answers : 1: $\theta_{12} = -\frac{1}{3}$ rad,

2: ~~δ_{212}~~ $\delta_{21} = 12$ mm

Maxwell's theorem:

In linearly elastic structures, the work done produced by load at point 1 in system 1 and relevant deformation caused by ~~point 2~~ load system 2 equals that work done produced by the load at ~~point 2~~ load system 2 & relevant deformation caused by load system 1 under position of load at system 2.



δ_{21} = Deflection caused by load P_2 in system 2 under position of load 'P_1'

δ_{12} = Deflection caused by load P_1 in system 1 under position of load P_2

Therefore,

$$P_1 \delta_{21} = P_2 \delta_{12}$$

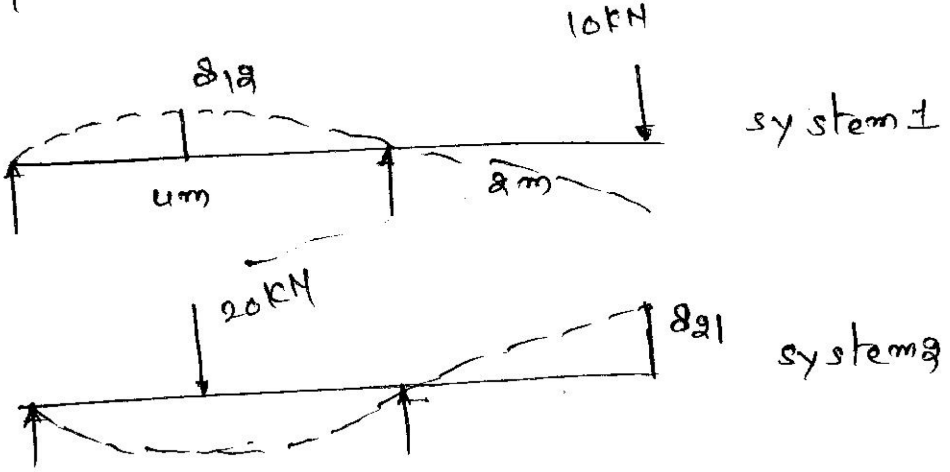
According to maxwell

$\boxed{P_1 = P_2}$ same loads are considered

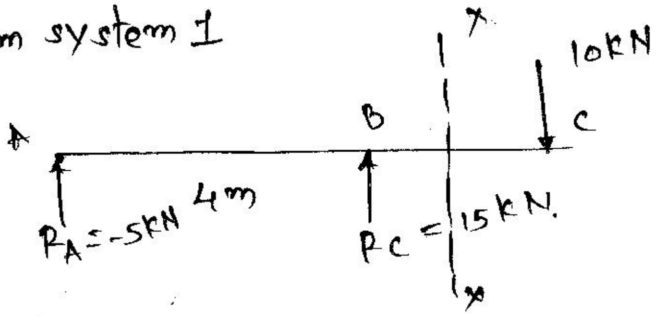
$$\therefore \boxed{\delta_{21} = \delta_{12}}$$

Maxwell's theorem can be proved with numerical example using double-integration method.

Example:



From system 1



$$M_x = R_A x + R_C(x-4)$$

$$M_x = -5x + 15(x-4)$$

$$EI\frac{d^2y}{dx^2} = 5x - 15(x-4)$$

$$EI\frac{dy}{dx} = \frac{5}{2}x^2 - 15\frac{(x-4)^2}{2} + C_1$$

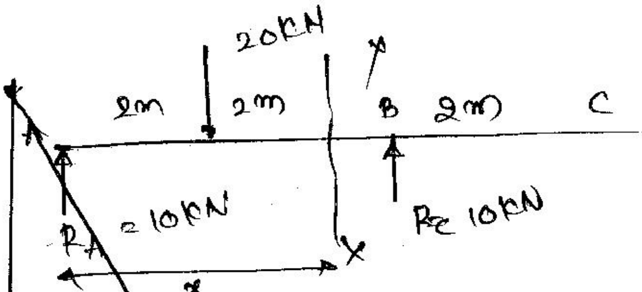
$$EIy = \frac{5}{2}\cdot\frac{x^3}{3} - \frac{15}{2}\cdot\frac{(x-4)^3}{3} + C_1x + C_2$$

put $x = 0; y = 0$; $C_2 = 0$ (-)ve power terms neglected

put $x = 4m$; $y = 0$

$$0 = \frac{5}{6}(4)^3 + 4C_1$$

$$\Rightarrow \boxed{C_1 = -\frac{40}{3}}$$



$$M_x = 10x - 20(x-2)$$

$$EI\frac{d^2y}{dx^2} = -10x + 20(x-2)$$

$$EI\frac{dy}{dx} = -\frac{10x^2}{2} + 20\frac{(x-2)^2}{2} + C_1$$

$$EIy = -\frac{10}{2}\frac{x^3}{3} + \frac{20}{2}\frac{(x-2)^3}{3} + C_1x + C_2$$

put $x = 0; y = 0$; $C_2 = 0$

put $x = 4m$; $y = 0$

$$0 = -\frac{10}{6}(4)^3 + \frac{20}{6}(2)^3 + C_1(4)$$

C.

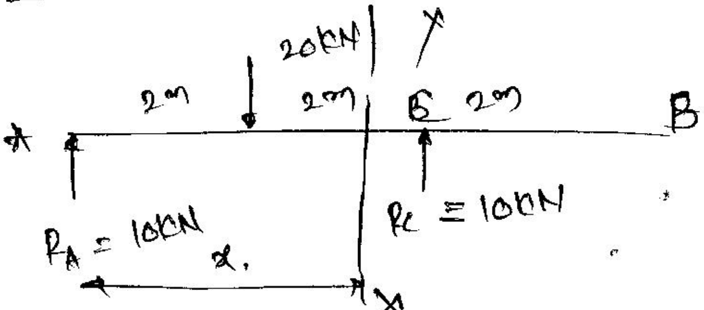
$\therefore EIy = \frac{5}{6}x^3 - \frac{15}{6}(x-4)^3 - \frac{40}{3}x$

put $x = 2m$; $y = \delta_{12}$

$EI\delta_{12} = \frac{5}{6}(2)^3 - \frac{40}{3}(2)$ power -ve terms neglected

$$\boxed{\delta_{12} = \frac{-20}{EI}}$$

system2:



$M_x = 10(4-x)$

$EI\frac{d^2y}{dx^2} = -10(4-x)$

$EI\frac{dy}{dx} = -10\frac{(4-x)^2}{2}\cdot\frac{1}{-1} + c_1$

$EIy = \frac{10}{2}\frac{(4-x)^3}{3}\cdot\frac{1}{-1} + c_1x + c_2$

put $x = 0$; $y = 0$;

$0 = -\frac{10}{6}(4)^3 + c_1(0) + c_2$

$c_2 = \frac{320}{3}$

put $x = 4m$; $y = 0$

$0 = 0 + c_1(4) + \frac{320}{3}$

$$\boxed{c_1 = -\frac{80}{3}}$$

$\therefore EIy = -\frac{10}{6}(4-x)^3 + \frac{-80}{3}x + \frac{320}{3}$

put $x = 6m$; $y = \delta_{21}$

$EI\delta_{21} = -\frac{10}{6}(4-6)^3 + \left(\frac{-80}{3}\times 6\right) + \frac{320}{3}$

$EI\delta_{21} = -40$

$$\boxed{\delta_{21} = \frac{-40}{EI}}$$

From theorem

$10\times\delta_{21} = 20\times\delta_{12}$

$10\times\left(\frac{-40}{EI}\right)$; $20\times\left(\frac{-20}{EI}\right)$

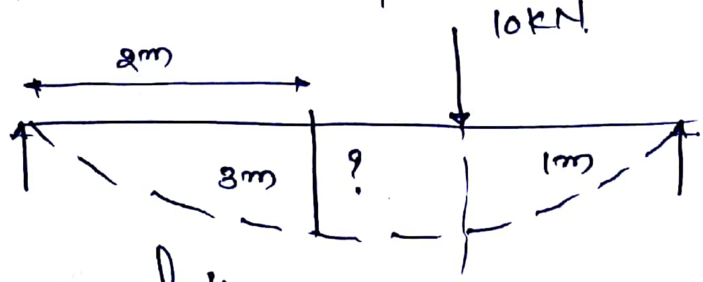
$\frac{-400}{EI}$; $\frac{-400}{EI}$

Hence it is proved.

Exercize problems:

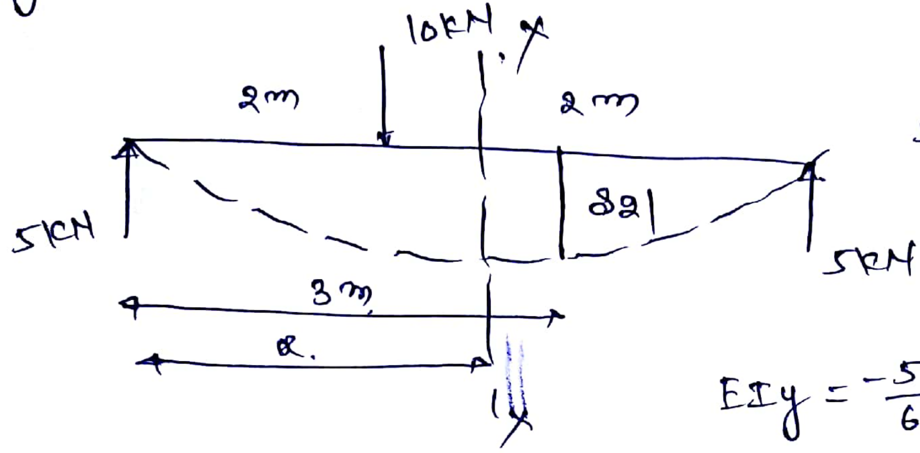
1. Determine the deflection at midspan, when the load 10kN is acting at 3m from the leftend in a S.S.B of 4m length. Using Maxwell's reciprocal theorem.

Sol:



solving usymmetrical loading is difficult

Using reciprocal theorem



symmetrical loading is easy.

$M_x = 5x - 10(x-2)$

$EI\frac{d^2y}{dx^2} = -5x + 10(x-2)$

$EI\frac{dy}{dx} = -\frac{5x^2}{2} + \frac{10(x-2)^2}{2} + C_1$

$EIy = -\frac{5x^3}{2 \cdot 3} + \frac{10}{2}\frac{(x-2)^3}{3} + C_1x + C_2$

neglecting (-)ve terms

put $x=0$; $y=0$; $C_2 = 0$

put $x = 4m$ $y=0$

$0 = \frac{-5}{6}(4)^3 + \frac{10}{6}(2)^3 + C_1(4)$

$0 = -40 + C_1 4 \Rightarrow C_1 = \frac{40}{4} = 10$

$\boxed{C_1 = 10}$

$EIy = -\frac{5}{6}x^3 + \frac{10}{6}(x-2)^3 + 10x$

put $x=3$

$EI\ \delta_{21} = -\frac{5}{6}(3)^3 + \frac{10}{6}(3-2)^3 + 10(3)$

$\boxed{\delta_{21} = \frac{55}{6EI}}$

From maxwell's reciprocal theorem

$\cancel{10} \times \delta_{21} = \cancel{10} \times \delta_{12}$

$\Rightarrow \boxed{\delta_{12} = \frac{55}{6EI}}$ ✓

Limitations:

1. Betti's & Maxwell's theorems are applicable for linear elastic structures.
2. Material of structure shall be homogenous.
3. These theorems are applicable for statically determinate & indeterminate structures also for calculating unknown deflections.
4. These theorems are used for only concentrated loads & moments.

astigliano's first theorem:

strain energy gets stored in a structural member that is strained due to imposed loads.

strain energy: An internal energy is stored due to external loads act on a body or structural member within the elastic limit.

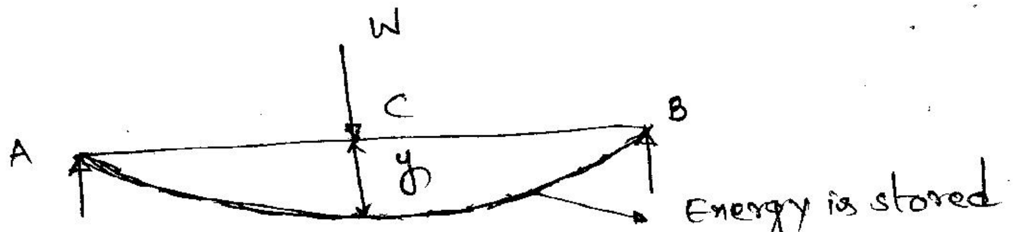
That energy is stored in the body during deformation process.

From equilibrium condition

Inernal Energy = External Energy

External energy is workdone by the forces or couples

$\boxed{\text{Energy : The capacity to do the work.}}$



W = Gradually applied load

y = deflection of point c / displacement of point C

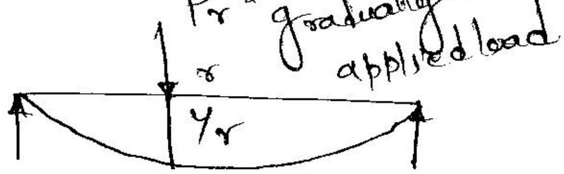
$W_e = \frac{1}{2} \times Wy$ = strain energy $\boxed{U = \frac{1}{2} Wy}$

slopes & deflections of a loaded member such as beams caused by its loading have relationships with the strain energy stored ~~there~~ in beams & can be determined by knowing the strain energy. In 1879, castigliano published two theorems, one of which pertains to the determination of slopes & defections of beams.

Castigliano's first theorem:

The partial derivative of the total external work done or the total internal energy stored w.r.t a load at a certain point of a statically determinate structure is equal to the deflection in the direction of the load at the point of its application.

$$\frac{\partial W_e}{\partial P_r} = y_r$$

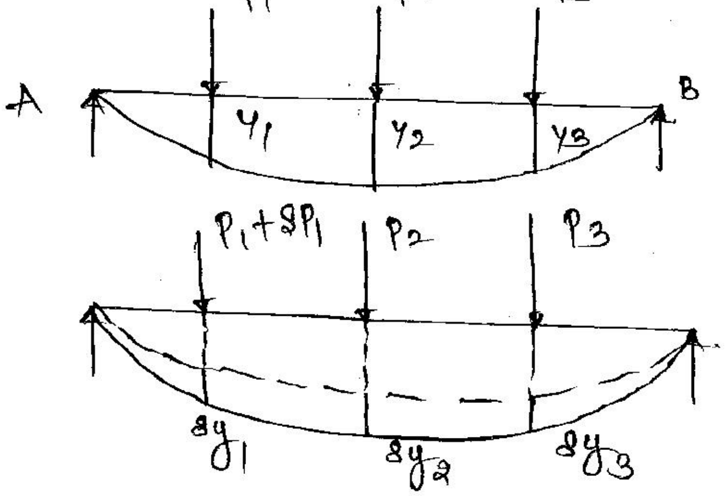


Consider a beam AB carrying gradually applied loads P_1, P_2 & P_3

Therefore, external work done on the beam is

$$W_e = \frac{1}{2} P_1 y_1 + \frac{1}{2} P_2 y_2 + \frac{1}{2} P_3 y_3$$

Now assume that a load δP_1 gradually applied to the load P_1 on the beam, so, the beam further deflects by δy_1, δy_2 & δy_3 at points 1, 2 & 3 respectively.



External work done

$$W_e = \frac{1}{2} P_1 y_1 + \frac{1}{2} P_2 y_2 + \frac{1}{2} P_3 y_3$$

Additional work done

$$\delta W_e = \left(P_1 + \frac{1}{2}\delta P_1\right)\delta y_1 + P_2 \delta y_2 + P_3 \delta y_3$$

For example.

$P_1 = 4\,kN$; $P_2 = 6\,kN$; $P_3 = 8\,kN$

$y_1 = 2\,mm$ $y_2 = 4\,mm$; $y_3 = 6\,mm$

add to P_1 by 2 kN (gradually applied)

$\delta y_1 = 1\,mm$; $\delta y_2 = 1.5\,mm$; $\delta y_3 = 2\,mm$

$(4 + \frac{1}{2} 2) 1 + (6 \times 1.5) + (8 \times 2)$

Therefore,

The total work done on the beam by the loads P_1 P_2 & P_3 and then addition of δP_1 is

$$W_e + \delta W_e = \frac{1}{2} P_1 y_1 + \frac{1}{2} P_2 y_2 + \frac{1}{2} P_3 y_3 + (P_1 + \frac{1}{2}\delta P_1)\delta y_1 + P_2 \delta y_2 + P_3 \delta y_3 \quad — (1)$$

But the total work done on the beam remains the same if $(P_1 + \delta P_1)$; P_2 & P_3 had been simultaneously but gradually applied.

$$W_e + \delta W_e = \frac{1}{2}(P_1 + \delta P_1)(y_1 + \delta y_1) + \frac{1}{2} P_2 (y_2 + \delta y_2) + \frac{1}{2} P_3 (y_3 + \delta y_3) \quad — (2)$$

$\therefore$ eq(1) = eq(2) $\Rightarrow$ $\cancel{\frac{1}{2} P_1 y_1} + \cancel{\frac{1}{2} P_2 y_2} + \cancel{\frac{1}{2} P_3 y_3} + P_1 \delta y_1 + \frac{1}{2}\delta P_1 \delta y_1$

$$+ P_2 \delta y_2 + P_3 \delta y_3 = \frac{1}{2}\left[\cancel{P_1 y_1}\right] + \frac{1}{2} P_1 \delta y_1 + \frac{1}{2}\delta P_1 y_1 + \frac{1}{2}\left[\delta P_1 \delta y_1\right]$$

$$+ \cancel{\frac{1}{2} P_2 y_2} + \frac{1}{2} P_2 \delta y_2 + \cancel{\frac{1}{2} P_3 y_3} + \frac{1}{2} P_3 \delta y_3$$

$$\Rightarrow P_1 \delta y_1 + \cancel{\frac{1}{2}\delta P_1 \delta y_1} + P_2 \delta y_2 + P_3 \delta y_3 = \frac{1}{2} P_1 \delta y_1 + \frac{1}{2}\delta P_1 y_1 + \cancel{\frac{1}{2}\delta P_1 \delta y_1} + \frac{1}{2} P_2 \delta y_2 + \frac{1}{2} P_3 \delta y_3$$

$$\Rightarrow \frac{1}{2} P_1 \delta y_1 + \frac{1}{2} P_2 \delta y_2 + \frac{1}{2} P_3 \delta y_3 = \frac{1}{2}\delta P_1 y_1$$

$$\Rightarrow P_1 \delta y_1 + P_2 \delta y_2 + P_3 \delta y_3 = \delta P_1 y_1$$

$$\delta W_e = P_1 \delta y_1 + \frac{1}{2}\delta P_1 \delta y_1 + P_2 \delta y_2 + P_3 \delta y_3$$

$$(\delta W_e - \frac{1}{2}\delta P_1 \delta y_1) = P_1 \delta y_1 + P_2 \delta y_2 + P_3 \delta y_3$$

$$\Rightarrow \delta W_e - \frac{1}{2}\delta P_1 \delta y_1 = \delta P_1 y_1$$

$$\Rightarrow \boxed{y_1 = \frac{\delta W_e}{\delta P_1}}.$$

since both δP_1 & δy_1 are very small quantities, their products can be neglected.

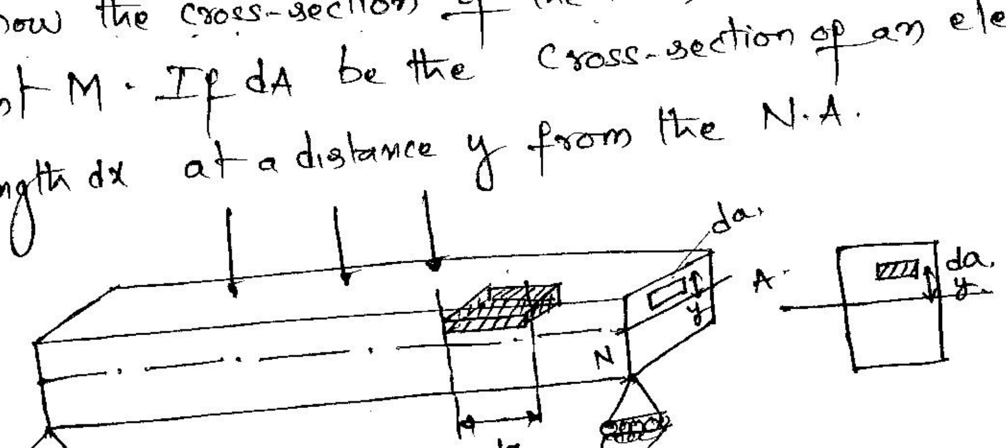
similarly $\boxed{\theta_1 = \frac{\delta W_e}{\delta M_1}}$

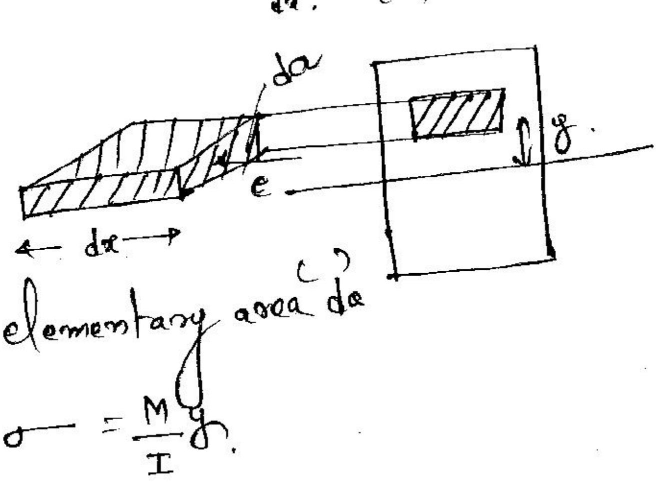
Briefly, when load changes either increases or decreases total strain energy also changes either increases or decreases respectively w.r.t. to the load. i.e., equal to deflection

Hence, it is proved.

Use of the above relationship in determining the slopes & deflections is known as the partial derivative method.

Let show the cross-section of the beam subjected to the bending moment M. If dA be the cross-section of an elementary beam of length dx at a distance y from the N.A.

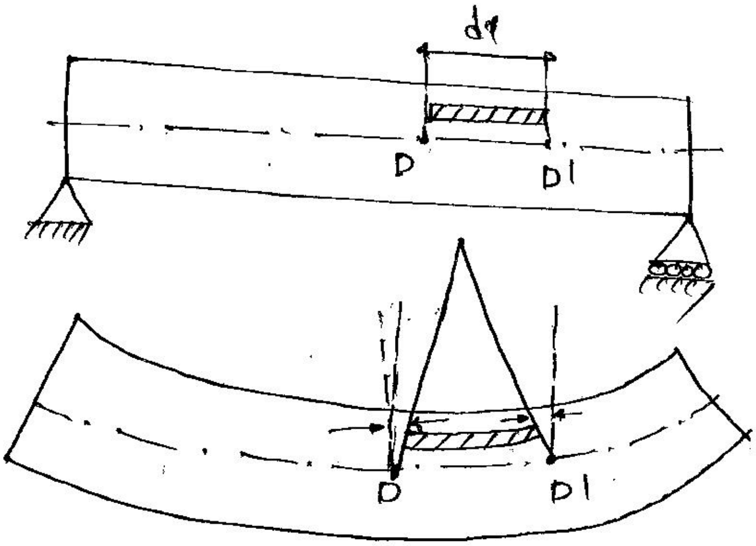


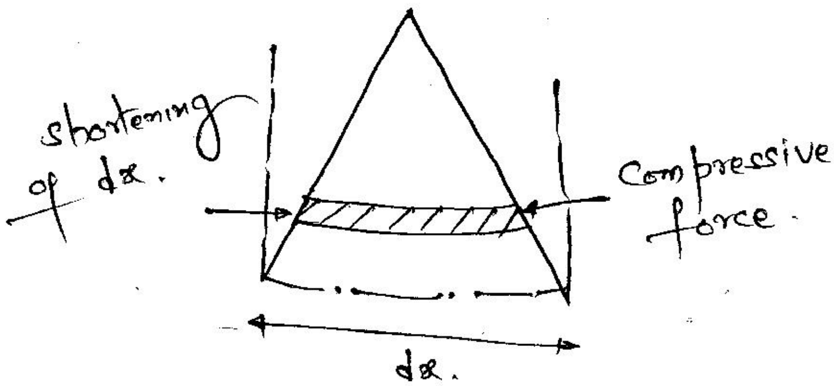


stress across the elementary area da

$$\sigma = \frac{M}{I} y.$$

Total force on the area $= \sigma \times da = \frac{M}{I} y\, da.$





Total shortening of length dx of the beam

$$= \frac{My}{EI}.dx \qquad \text{strain} = \frac{\text{change in length}}{\text{original length}}$$

Therefore, Energy stored = $\frac{1}{2}$ Total force × shortening

shortening of length = strain × dx

$$= \frac{1}{2}\left(\frac{My}{I}da\right)\left(\frac{My}{EI}dx\right) \qquad \epsilon = \frac{\sigma}{E} = \frac{My}{EI}$$

∴ Total energy stored in the beam

$$= \int_0^l \int_0^A \frac{My}{I} da \, \frac{My}{EI} dx.$$

$$= \int_0^l \int_0^A \frac{M^2}{EI^2} y^2 da \, dx.$$

But $\int_0^A y^2 dA = I$

$$U = \frac{1}{2}\int_0^l \frac{M^2}{EI^2} \times I\,dx = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx.$$

Since by the principle of conservation of energy, the total external workdone is equal to the total internal energy stored.

$$U = W_e = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx.$$

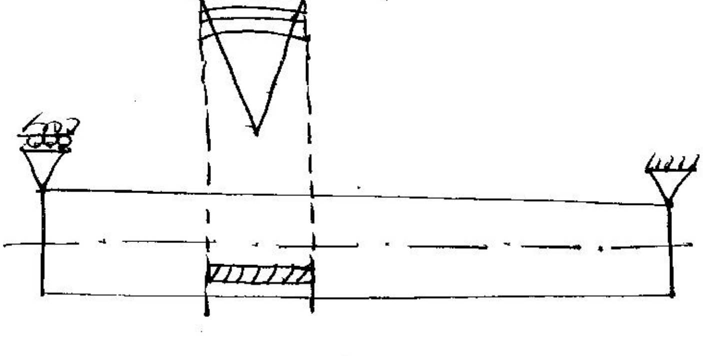
$$\frac{dW_e}{dP_1} = \frac{d}{dP_1}\left[\frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx\right] = \int_0^l \frac{M \times \frac{dM}{dP_1}}{EI}\,dx$$

From the equation

$$\boxed{y_1 = \frac{dW_e}{dP_1} = \frac{1}{EI}\int_0^l M \cdot \frac{dM}{dP_1}\,dx.}$$

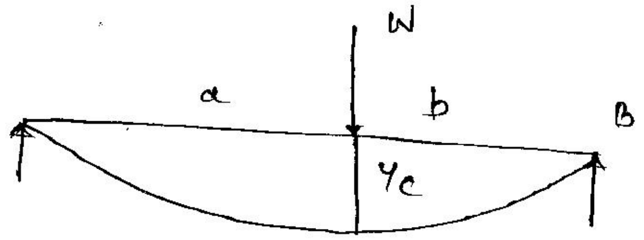
lll'y

$$\boxed{\theta_1 = \frac{1}{EI}\int_0^l M \times \frac{dM}{dM_1} \times dx.}$$



Exercize problems:

1. Using castigliano's theorem, calculate the deflection at point of application of load as shown in fig. Take EI is same through the span.



Sol: As per the castigliano's theorem followed by strain energy due to bending, deflection at point of application of load P_1 is $\frac{dW_e}{dP_1}$

$$y_1 = \frac{dW_e}{dP_1} \qquad W_e = U = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx$$

In this example, $y_1 = y_c$; $P_1 = W$

As per castigliano's theorem

$$y_c = \frac{dW_e}{dW} = \frac{d\left[\frac{1}{2}\int_0^l \frac{M^2}{EI}dx\right]}{dW}$$

$$\Rightarrow \quad y_c = \frac{1}{\cancel{2}}\frac{1}{EI}\int_0^l \cancel{2}M\cdot\frac{dM}{dW}\,dx$$

$$y_c = \frac{1}{EI}\int_0^l M\frac{dM}{dW}\,dx.$$

Bending moments at section x-x within portions AC & CB

For portion AC $M_x = \frac{Wbx}{l} \Rightarrow; \quad \frac{dM_x}{dW} = \frac{bx}{l}$

For portion CB $M_x = \frac{Wa}{l}x. \quad \frac{dM_x}{dW} = \frac{ax}{l}$

$$\therefore \text{Total deflection } y_c = \frac{1}{EI_1}\int_0^a \frac{Wbx}{l}\cdot\frac{bx}{l}\,dx + \frac{1}{EI}\int_0^b \frac{Wax}{l}\cdot\frac{ax}{l}\,dx.$$

$$y_c = \frac{1}{EI}\frac{Wb^2}{l^2}\int_0^a x^2\,dx \quad + \frac{1}{EI}\frac{Wa^2}{l^2}\int_0^b x^2\,dx.$$

$$y_c = \frac{1}{EI}\frac{Wb^2}{l^2}\left[\frac{x^3}{3}\right]_0^a + \frac{1}{EI}\frac{Wa^2}{l^2}\left[\frac{x^3}{3}\right]_0^b$$

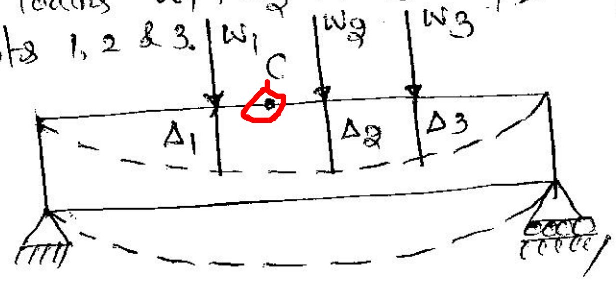
$$= \frac{1}{EI}\frac{Wb^2}{l^2}\left(\frac{a^3}{3}\right) + \frac{1}{EI}\frac{Wa^2}{l^2}\left(\frac{b^3}{3}\right)$$

$$y_c = \frac{Wa^2b^2}{3l^2EI}\left[a+b\right] = \frac{Wa^2b^2}{3\cancel{l^2}EI}\cancel{l}$$

$$\boxed{y_c = \frac{Wa^2b^2}{3EIl}}$$

Unit load method:

step (i): Consider a simply supported beam subject to a gradually applied loads W_1, W_2 & W_3 respectively as shown in fig at points 1, 2 & 3.

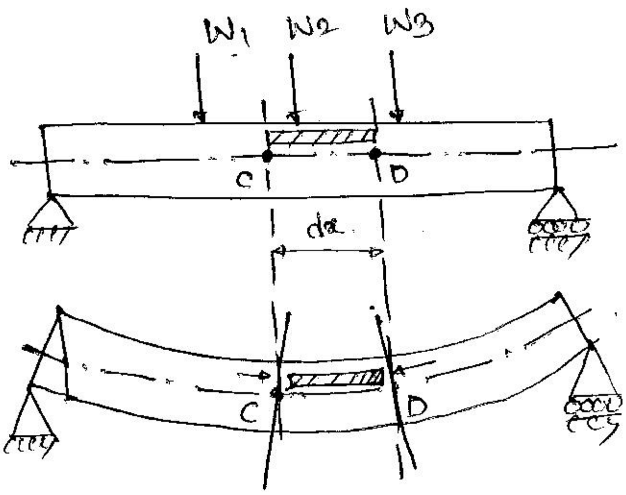


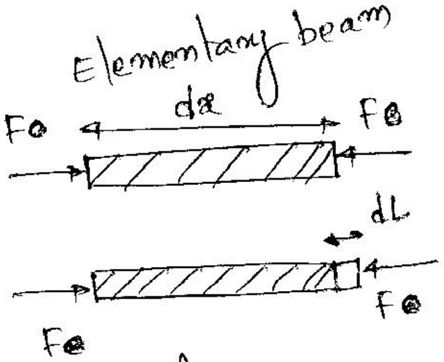
Let C be the point on the beam, Let Δ_C be the deflection at point C. (Unit load method is used to calculate deflection at any point on the beam)

From fig.

$$\text{External work done} = \frac{1}{2} W_1 \Delta_1 + \frac{1}{2} W_2 \Delta_2 + \frac{1}{2} W_3 \Delta_3$$

$$\text{Internal energy} = \frac{1}{2} \Sigma F \cdot dL$$





Internal energy of elementary beam $= \frac{1}{2} F \times dL$

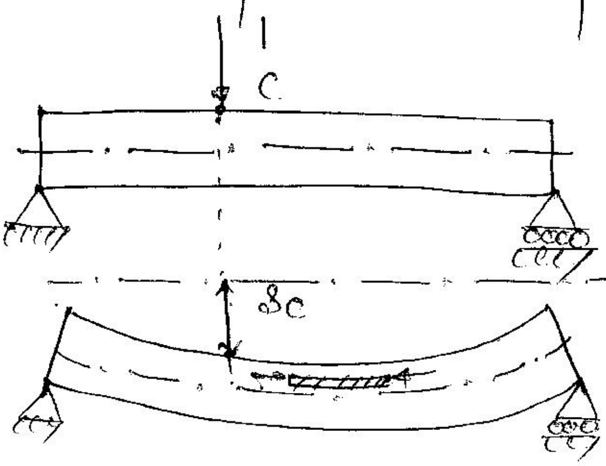
$$\text{Total internal energy} = \frac{1}{2} \Sigma F \cdot dL$$

$\boxed{dL = \text{shortening of element beam}}$

for equilibrium state,

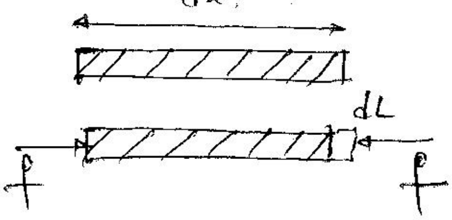
$$\frac{1}{2} W_1 \Delta_1 + \frac{1}{2} W_2 \Delta_2 + \frac{1}{2} W_3 \Delta_3 = \frac{1}{2} \Sigma F \cdot dL \quad \text{---(1)}$$

step(2): Apply unit load of '1' instead of W_1, W_2 & W_3



External work done $= \frac{1}{2} \times 1 \times \delta_C$

Internal energy $= \frac{1}{2} \Sigma f \cdot dL$



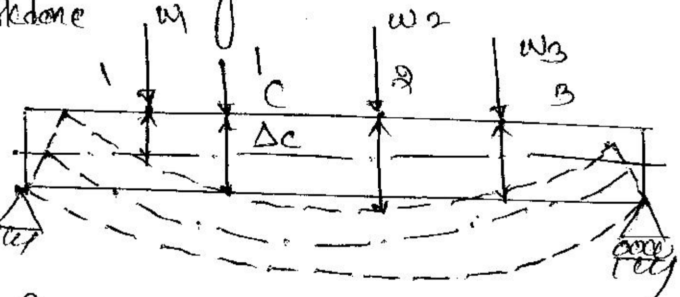
From equilibrium state

$$\frac{1}{2} \times 1 \times \delta_C = \frac{1}{2} \Sigma f \cdot dL \quad \text{--- (2)}$$

f = force of an elementary area due to unit load

step(3): Apply W_1 W_2 & W_3 gradually applied loads on the beam including unit load.

Additional workdone



Additional External workdone $= \frac{1}{2} W_1 \times \Delta_1 + 1 \times \Delta_C + \frac{1}{2} W_2 \Delta_2 + \frac{1}{2} W_3 \Delta_3$

(settled load)

Additional Internal energy $= \frac{1}{2} \Sigma F \cdot dL + \Sigma f \cdot dL$

(settled load)

From equilibrium state

$$\frac{1}{2} W_1 \Delta_1 + \frac{1}{2} W_2 \Delta_2 + \frac{1}{2} W_3 \Delta_3 + 1 \cdot \Delta_C = \frac{1}{2} \Sigma F \cdot dL + \Sigma f \cdot dL \quad \text{--- (3)}$$

$\therefore$ Total work done. ~~sum~~ add (2) & (3) equations

$$\left.\begin{aligned}&\frac{1}{2}W_1\Delta_1 + \frac{1}{2}W_2\Delta_2 + \frac{1}{2}W_3\Delta_3 + \frac{1}{2}\cdot 1\cdot \delta_c + 1\cdot\Delta_c \\ &= \frac{1}{2}\Sigma F\,dL + \Sigma f\cdot dL + \frac{1}{2}\Sigma f\cdot dL \quad \text{---(4)}\end{aligned}\right\} \begin{array}{l}\text{Workdone by unit load}\\ \text{addition \& +}\\ \text{workdone by}\\ \text{gradually applied loads}\end{array}$$

From eq(1) & eq(2)

eq(1) ; workdone by gradually applied loads

eq(2) ; Workdone by unit load

$$eq(1) + eq(2) \Rightarrow \frac{1}{2}W_1\Delta_1 + \frac{1}{2}W_2\Delta_2 + \frac{1}{2}W_3\Delta_3 + \frac{1}{2}\times 1\cdot\delta_c$$

$$= \frac{1}{2}\Sigma F\cdot dL + \frac{1}{2}\Sigma f\cdot dL \quad \text{---(5)}$$

~~equating (4) & (5)~~

subtracting eq(4) − eq(5)

$$\cancel{\frac{1}{2}W_1\Delta_1} + \cancel{\frac{1}{2}W_2\Delta_2} + \cancel{\frac{1}{2}W_3\Delta_3} + \cancel{\frac{1}{2}\times 1\cdot\delta_c} + 1\cdot\Delta_c - \left(\cancel{\frac{1}{2}W_1\Delta_1} + \cancel{\frac{1}{2}W_2\Delta_2} + \cancel{\frac{1}{2}W_3\Delta_3} + \cancel{\frac{1}{2}\times 1\cdot\delta_c}\right) = \cancel{\frac{1}{2}\Sigma F\,dL} + \Sigma f\cdot dL + \cancel{\frac{1}{2}\Sigma f\cdot dL} - \left(\cancel{\frac{1}{2}\Sigma F\,dL} + \cancel{\frac{1}{2}\Sigma f\cdot dL}\right)$$

$$\boxed{1\cdot\Delta_c = \Sigma f\cdot dL}$$

Here f = force of an elementary area (da) due to unit load

$$\therefore f = \sigma' da \qquad \sigma' = \frac{my}{I}$$

$$= \frac{my}{I}da.$$

dL = shortening of elementary beam (dx) due to external loads

$$dL = \epsilon \times dx.$$

$$\epsilon = \frac{\sigma}{E} = \frac{My}{EI}$$

$$= \epsilon \times dx$$

$$= \frac{My}{EI}dx.$$

Applying integration

$$1.\ \Delta_C = \int_0^L \int_0^{da} \sigma' \, da \times \epsilon \times dx.$$

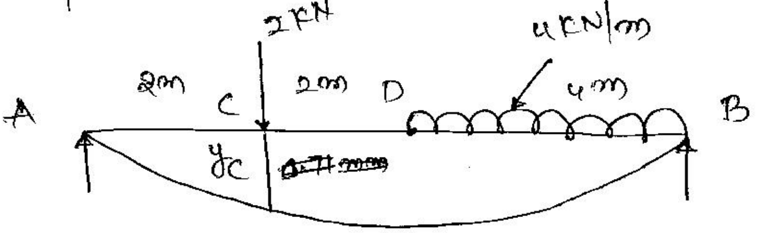
$$= \int_0^L \int_0^{da} \frac{my}{I} \cdot \frac{My}{EI} \cdot da \, dx.$$

$$\Delta_C = \int_0^L \frac{mM}{EI^{\cancel{2}}} \cancel{I} \cdot dx \qquad \text{Here } \int_0^{da} y^2 da = I$$

$$\boxed{\Delta_C = \int_0^L \frac{Mm}{EI} dx}$$

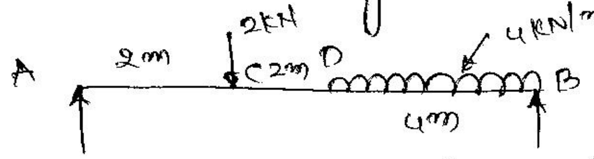
Exercize problems:

1.



Determine deflection at midspan C

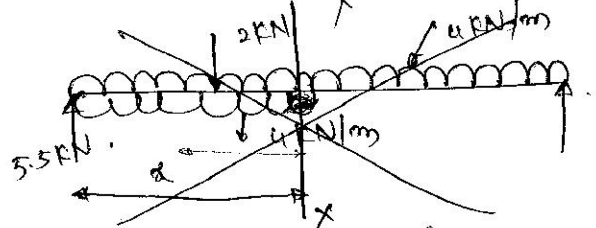
Sol: Double integration method.



Reactions $(R_A \times 8) - (2\times 6) - (4\times 4\times \frac{4}{2}) = 0$

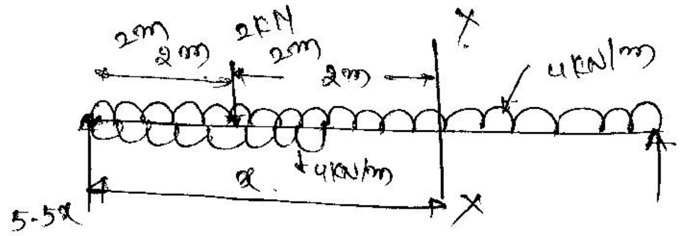
$8R_A - 12 - 32 = 0 \Rightarrow R_A = 5.5\text{ KN}$

$R_B = 2 + 16 - 5.5 = 12.5\text{ KN}$



$4\times 4(x-2)$

$M_x = \cancel{5.5x + 4\times\frac{x^2}{2}}$



$M_x = 5.5x + 4\times 4(x-2) - 4\times\frac{x^2}{2} - 2(x-2)$

$M_x = 5.5x + 16(x-2) - 2x^2 - 2(x-2)$

$EI\frac{d^2y}{dx^2} = -5.5x - 16(x-2) + 2x^2 + 2(x-2)$

~~put~~ integrating

$EI\frac{dy}{dx} = -5.5\frac{x^2}{2} - 16\frac{(x-2)^2}{2} + \frac{2x^3}{3} + 2\frac{(x-2)^2}{2} + C_1$

$EIy = -\frac{5.5}{2}\frac{x^3}{3} - \frac{8(x-2)^3}{3} + \frac{2}{3}\frac{x^4}{4} + \frac{(x-2)^3}{3} + C_1x + C_2$

Put $x = 0$ $\quad C_2 = 0$

Put $x = 8m$

$0 = -\frac{5.5}{6}\times 8^3 - \frac{8}{3}\times 6^3 + \frac{2}{12}\times 6^4 + \frac{6^3}{3} + C_1(8)$

$0 = -\frac{2272}{3} + 8C_1$

$C_1 = \frac{284}{3} = 94.67$

Put $x = 2$

$EIy_C = -\frac{5.5}{6}\times 2^3 - 0 + \frac{2}{12}(2)^4 + 0 + 94.67$

$$\boxed{y_C = \frac{90.00}{EI}}$$

$I = 1.33\times 10^{-3}\ m^4$

$E = 21.718\ \frac{KN}{mm^2}$

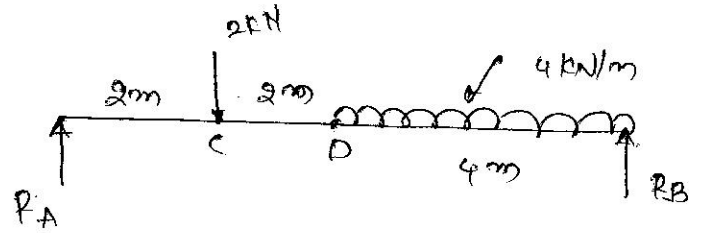
$= 21.718\ \frac{KN}{10^{-6} m^2}$

$= 21.718\times 10^{6}\ KN/m^2$

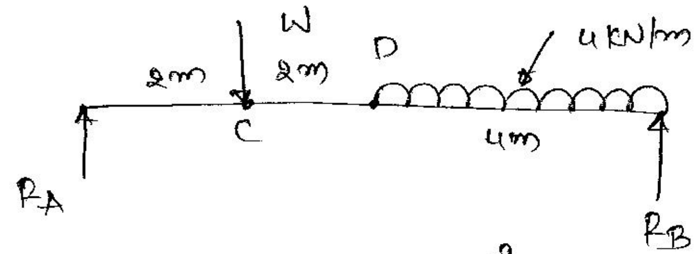
$EI = 28887.6\ KN.m^2$

$y_C = \frac{90}{28887.6} = 3.11\times 10^{-3} m = 3.11\ mm$

Ex 01: Using Castigliano's first theorem..



For calculating deflection at point C, a dummy load of 'W' is placed where deflection is to be calculated & remove ~~load~~ if any load acts on it



$$(R_A \times 8) - (W \times 6) - \left(4 \times 4 \times \frac{4}{2}\right) = 0$$

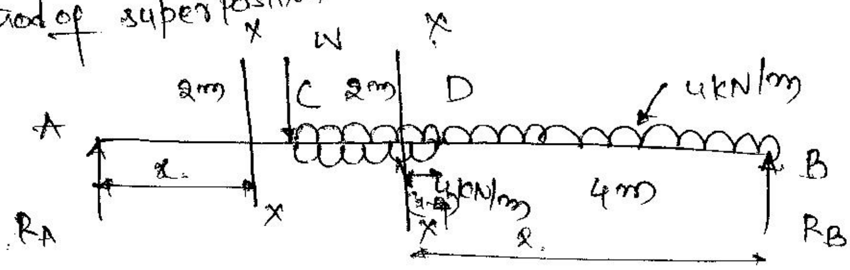
$$8R_A - 6W - 32 = 0 \Rightarrow R_A = \frac{6W+32}{8}$$

$$\boxed{R_A = \frac{3W+16}{4}}$$

$$R_B = (16 + W) - \frac{3W+16}{4} = \frac{64 + 4W - 3W - 16}{4} = \frac{W+48}{4}$$

$$\boxed{R_B = \frac{W+48}{4}}$$

Using method of superposition



Portion	origin	limits	M_x	$\frac{\partial M_x}{\partial W}$
AC	A	0 – 2	$\left(\frac{3W+16}{4}\right)x$	$\frac{1}{4}(3) = 0.75x$.
BC	B	0 – 6	$\left(\frac{W+48}{4}\right)x - \frac{4x^2}{2} + 4\frac{(x-4)^2}{2}$	$\frac{1}{4}x = 0.25x$.

Portion AC:

$$y_{C(AC)} = \frac{1}{EI}\int_0^2 M_x \cdot \frac{dM_x}{dW}\, dx.$$

$$= \frac{1}{EI}\int_0^2 (0.75W + 4)x \times 0.75x\, dx.$$

$$= \frac{1}{EI}\int_0^2 (0.56W + 3)x^2\, dx$$

$$= \frac{1}{EI}\left[0.56W\frac{x^3}{3} + \cancel{3}\frac{x^3}{\cancel{3}}\right]_0^2 = \frac{1}{EI}\left[1.49W + 2.67\right]$$

put $W = 2\,kN \Rightarrow y_C = \frac{1}{EI}\left[5.65\right]$

Portion CB:

$$y_C = \frac{1}{EI}\int_0^6 \left[(0.25W + 12)x - 2x^2 + 2(x-4)^2\right] \times 0.25x\, dx.$$

$$= \frac{1}{EI}\int_0^6 \left[0.25Wx + 12x - \cancel{2x^2} + \cancel{2x^2} - 16x + 32\right] \times 0.25x\, dx.$$

$$= \frac{1}{EI}\int_0^6 \left[0.25Wx - 4x + 32\right] \times 0.25x\, dx.$$

$$= \frac{1}{EI}\int_0^6 (0.0625Wx^2 - x^2 + 8x)\, dx$$

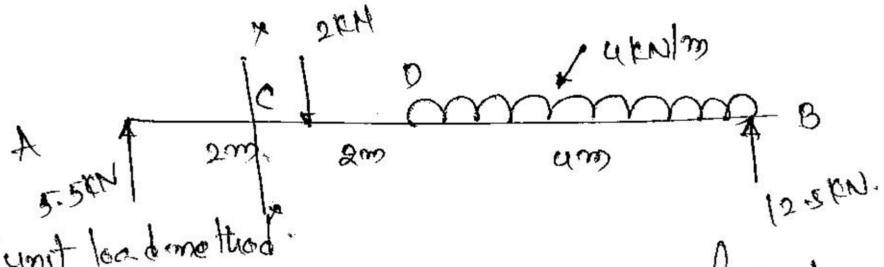
$$= \frac{1}{EI}\left[0.0625W\frac{x^3}{3} - \frac{x^3}{3} + 8\frac{x^2}{2}\right]_0^6 \qquad \text{put } W = 2\,kN.$$

$$= \frac{1}{EI}\left[9 - 72 + 144\right]$$

$$y_{C(CB)} = \frac{81}{EI}.$$

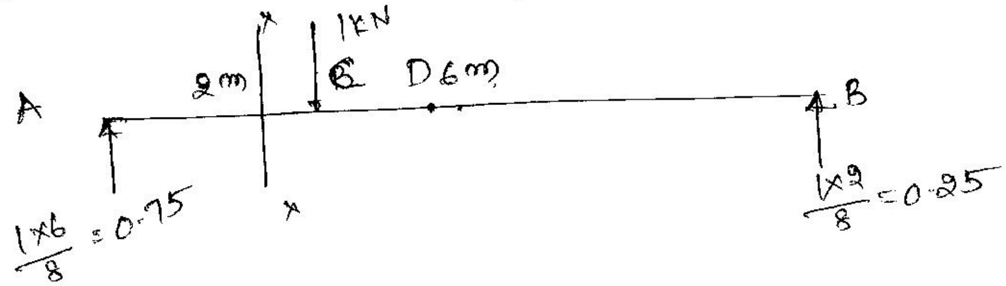
Final deflection $y_C = \frac{5.65}{EI} + \frac{81}{EI} = \frac{86.65}{EI}.$ ✓

Unit load method:



Using 'unit load method'.

step (i): Apply unit load instead of extonal loading



Origin	Portion	limits	EI	M	m
A	AC	0 - 2	EI	$5.5x$	$0.75x$
C	CD	0 - 2	EI	$5.5(x+2) - 2x$	$0.75(x+2) - 1.x$
B	DB	0 - 4	EI	$12.5x - \frac{4x^2}{2}$	$0.25x$

$$\Delta_C = \int_0^2 \frac{5.5x \times 0.75x}{EI}\,dx + \int_0^2 \frac{[5.5(x+2)-2x][0.75(x+2)-x]}{EI}\,dx$$

$$+ \int_0^4 \frac{(12.5x - 2x^2)\,0.25x}{EI}\,dx$$

Portion AC:

$$y_{C(AC)} \quad \frac{4.125}{EI} \times \left[\frac{x^3}{3}\right]_0^2 = \frac{11}{EI}$$

Portion CD

$$\int_0^2 \frac{(5.5x + 11 - 2x)(0.75x + 1.5 - x)}{EI}\,dx$$

$$\int_0^2 \frac{(3.5x + 11)(1.5 - 0.25x)}{EI}\,dx$$

$$= \int_0^2 \frac{(5.25x - 0.875x^2 + 16.5 - 2.75x)}{EI}\,dx$$

$$\frac{1}{EI}\int_0^2 (2.5x - 0.875x^2 + 16.5)\,dx = \frac{1}{EI}\left[2.5\frac{x^2}{2} - 0.875\frac{x^3}{3} + 16.5x\right]_0^2$$

$$= \frac{1}{EI}\left[35.67\right]$$

Portion OB:

$$\frac{1}{\cancel{EI}}\int_0^4 \frac{(12.5x - 2x^2)\times 0.25x}{EI}$$

$$\int_0^4 \frac{3.125x^2 - 0.5x^3}{EI}\,dx.$$

$$= \frac{1}{EI}\left[3.125\frac{x^3}{3} - 0.5\frac{x^4}{4}\right]_0^4$$

$$= \frac{1}{EI}\left[34.67\right]$$

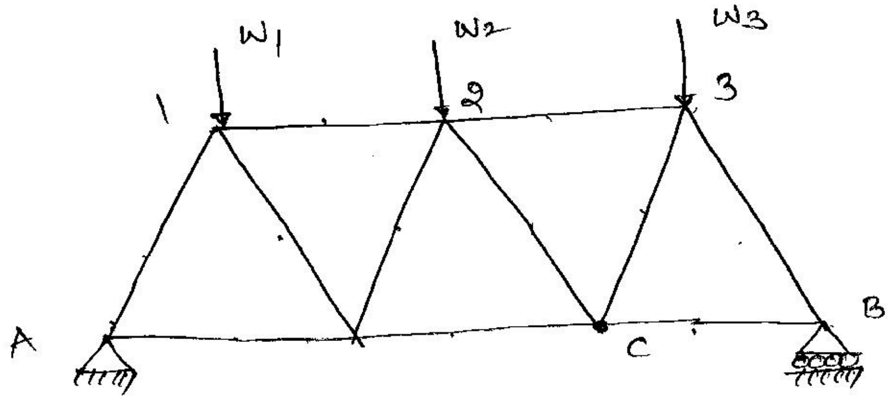
Final deflection

$$\Delta_C = \frac{11}{EI} + \frac{35.67}{EI} + \frac{34.67}{EI}$$

$$\boxed{\Delta_C = \frac{81.34}{EI}}$$

Unit load method: (Pin Jointed frames)

step(i): Consider a statically determinate truss subject to a gradually applied loads W_1 W_2 & W_3 at the joints 1, 2 & 3 respectively, as shown in fig.



$D_s = nm + r - yj$

$= 1(11) + 3 - 2(7)$

$= 14 - 14 = 0$

$D_s = 0$

statically determinate.

Let 'y_c' be the deflection at point C. let $P_1, P_2, P_3 \ldots$ etc. be the member forces due to external load system.

$\therefore$ Total external workdone $W_e = \Sigma \frac{P_i^2 L_i}{2 A_i E}$

Here, each member of truss experiences the axial force either compression or tension.

Work done of each member $= \frac{1}{2} P \times dL$

Here dL = strain x original length.

$\epsilon = \frac{\sigma}{E}$

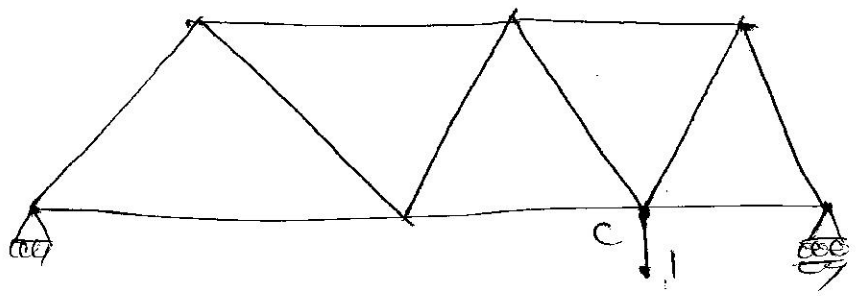
$\epsilon = \frac{1}{E} \cdot \frac{P}{A}$

$W_e = \frac{1}{2} \cdot P \cdot dL = \frac{1}{2} P \cdot \frac{1}{E} \frac{P}{A} \times l$.

$W_e = \frac{P^2 l}{2AE}$ for each member.

$\therefore$ Total Workdone = Total energy stored.

step(ii) : Let the deflection caused by unit load at C be δy_c



Here k_1, k_2, k_3 ----- be the member forces due to unit load at C.

$$\text{Workdone} = \frac{1}{2} \cdot 1 \cdot \delta y_c$$

$$\text{strain energy stored} = \Sigma \frac{k_i^2 l_i}{2A_iE}$$

As the structure is carrying unit load at C. let the given load system be applied on the structure.

The forces in the members of the structure (P_1+K_1) (P_2+k_2) ....

$$\therefore \text{Total work done} = \underbrace{W_e}_{\text{Additional work done by external load system}} + \underbrace{\frac{1}{2} 1 \cdot \delta y_c}_{\text{work done by unit load}} + \underbrace{1 \cdot y_c}_{\text{Additional work done by external load system at point C.}}$$

$$\text{Total strain energy stored} = \Sigma \frac{(P_i+K_i)^2 l_i}{2A_iE.}$$

$$\therefore W_e + \frac{1}{2} 1 \cdot \delta y_c + 1 \cdot y_c = \Sigma \frac{P_i^2 l_i}{2A_iE.} + \Sigma \frac{k_i^2 l_i}{2A_iE} + \Sigma \frac{\cancel{2} P_i k_i l_i}{\cancel{2} A_iE.}$$

$$\cancel{\Sigma \frac{P_i^2 l_i}{2A_iE}} + \cancel{\Sigma \frac{k_i^2 l_i}{2A_iE}} + 1 \cdot y_c = \cancel{\Sigma \frac{P_i^2 l_i}{2A_iE}} + \cancel{\Sigma \frac{k_i^2 l_i}{2A_iE}} + \Sigma \frac{P_i k_i l_i}{A_iE}$$

$$\boxed{y_c = \Sigma \frac{P_i k_i l_i}{A_iE.}}$$

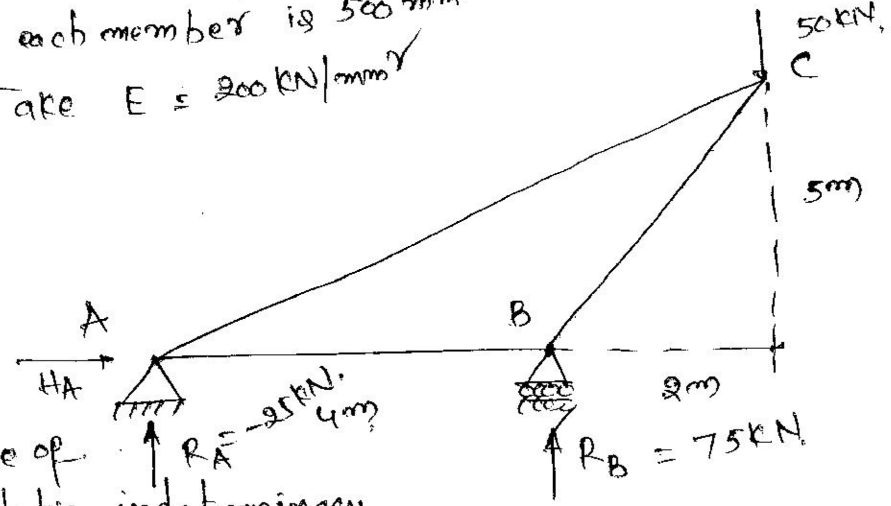
~~From eq(2) & (3)~~ ; add eq(2) & (3)

~~$4N_c + 1 \cdot y_c + \frac{1}{2} 1.5 y_c =$~~

1. Determine the vertical deflection at the point of application of the load of a plane steel truss supported as shown in fig. Assume cross sectional area of each member is 500 mm²

Take $E = 200\ kN/mm^2$

Sol:



step(i): Degree of static indeterminacy

$$D_s = n\,m + r - 2j = 1(3) + 3 - 2(3) = 6 - 6 = 0$$

statically determinate structure

step(ii):

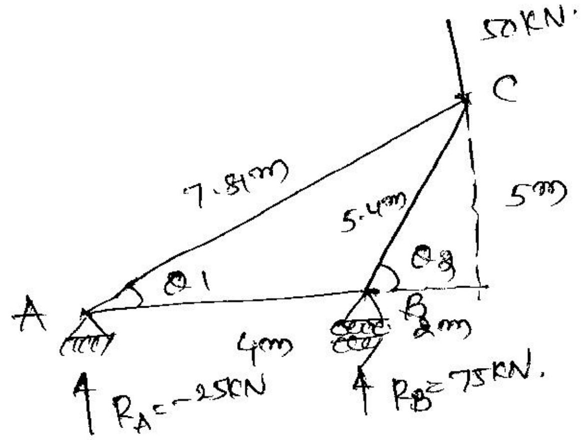
$$\Sigma H = 0 \Rightarrow H_A = 0$$

Taking moments about 'B'

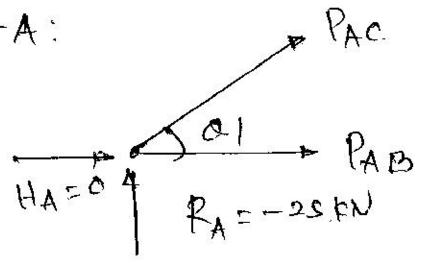
$$R_A \times 4 + (50 \times 2) = 0 \Rightarrow 4R_A = -100 \Rightarrow \boxed{R_A = -25\ kN}$$

$$R_B = 50 + 25 = 75\ kN \quad (R_B = 75\ kN)$$

(iii): By using method of joints; Assume all member forces are in tension & taken as + ve.



Joint A:



$\Sigma H = 0 \Rightarrow \quad H_A + P_{AB} + P_{AC}\cos\theta_1 = 0$

$\Sigma V = 0 \Rightarrow \quad P_{AC}\sin\theta_1 + R_A = 0$

$\sin\theta_1 = \frac{5}{7.81} = 0.64$

$\cos\theta_1 = \frac{6}{7.81} = 0.77$

$P_{AC}(0.64) + (-25) = 0$

$P_{AC} = \frac{25}{0.64} = 39.0$ kN.

$\boxed{P_{AC} = 39 \text{ kN}}$ (T)

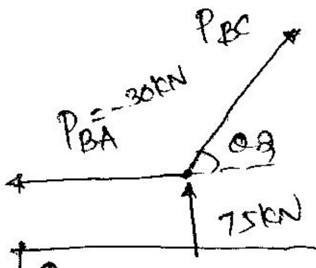
$\therefore \quad 0 + P_{AB} + 39(0.77) = 0$

$\boxed{P_{AB} = -30 \text{ kN}}$ (C)

$\sin\theta_2 = \frac{5}{5.4} = 0.93$

$\cos\theta_2 = \frac{2}{5.4} = 0.37$

Joint B:



$\Sigma H = 0 \Rightarrow$

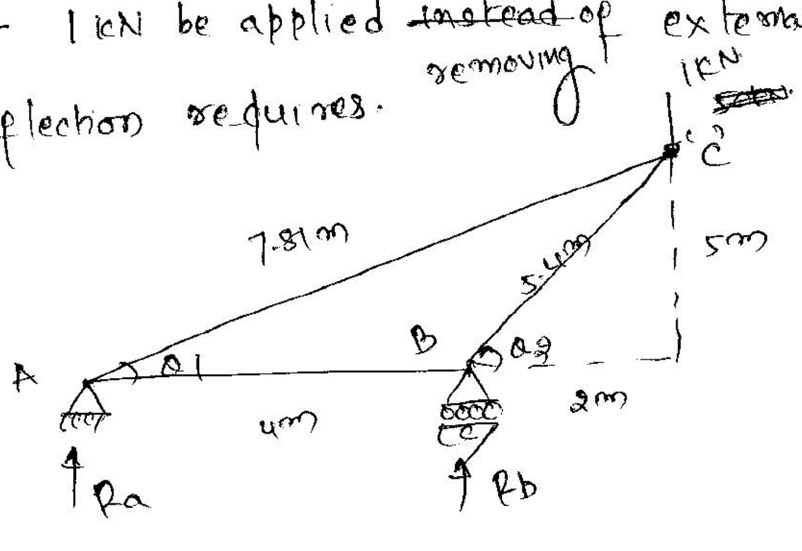
$-P_{BA} + P_{BC}\cos\theta_2 = 0$

$\Rightarrow \quad -(-30) + P_{BC}(0.37) = 0$

$\boxed{P_{BC} = -81.0 \text{ kN}}$ (C)

$P_{BC} = \frac{-30}{0.37} = -81.0$ kN.

step (iv): Let 1 kN be applied by ~~replacing~~ ~~instead~~ removing of external loads where deflection requires.



$\Sigma H = 0 \Rightarrow H_a = 0$

& Taking moments about A

$$(R_a \times 4) + (1 \times 2) = 0$$

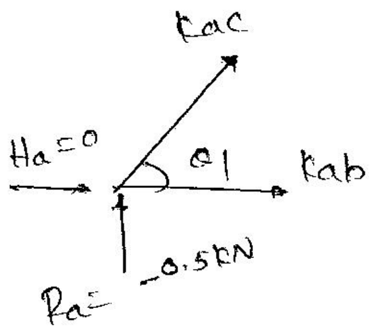
$$R_a = \frac{-2}{4} = -0.5\,kN.$$

$$R_B = 1 - R_a = 1 - (-0.5)$$

$$= 1.5\,kN.$$

Using method of joints; Assume all member forces are in tension & taken as (+)ve.

Joint A:



$\Sigma H = 0 \Rightarrow H_a + K_{ac}\cos\theta_1 + K_{ab} = 0$

$\Sigma V = 0 \Rightarrow K_{ac}\sin\theta_1 + R_a = 0$

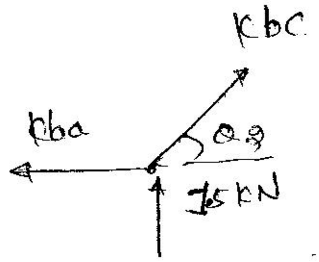
$K_{ac}(0.64) + (-0.5) = 0$

$\boxed{K_{ac} = 0.78\,kN}$ (T)

$\therefore\ 0 + (0.78 \times 0.77) + K_{ab} = 0$

$\Rightarrow \boxed{K_{ab} = -0.6\,kN}$ (C)

Joint B:



$\Sigma V = 0$

$\Rightarrow K_{bc}\sin\theta_2 + 1.5 = 0$

$K_{bc} = \frac{-1.5}{0.93} = -1.61$

$\boxed{K_{bc} = -1.61\,kN}$ (C)

$E = 200\,kN/mm^2$

From the table

Member	P_i (kN)	K_i (kN)	L_i (mm)	A_i (mm^2)	$\frac{P_i K_i L_i}{A_i E}$
AB	-30	-0.6	4000	500	$+0.78$
BC	-81	-1.61	5400	500	$+7.048$
AC	$+39$	$+0.78$	7810	500	$+2.37$
					10.137 mm

$\therefore \boxed{y_c = 10.137\ mm}$